Supplementary Material for "Local Indirect Least Squares and Average Marginal Effects in Nonseparable Structural Systems,"
by Schennach, S. M., H. White, and K. Chalak.

Proof of Lemma 3.1. This result holds by construction, using integration by parts under Assumption 3.3. ■

Lemma A.5 *Suppose Assumption 3.4 holds. Then* $\sup_{z\in\mathbb{R}}\left|k^{(\lambda)}(z)\right|<\infty$, $\int\left|k^{(\lambda)}(z)\right|dz<\infty$, $0<\int\left|k^{(\lambda)}(z)\right|^2 dz<\infty$, $\int\left|k^{(\lambda)}(z)\right|^{2+\delta}dz<\infty$, *and* $|z|\left|k^{(\lambda)}(z)\right|\to 0$ *as* $|z|\to\infty$.

Proof. The Fourier transform of $k^{(\lambda)}(z)$ is $(-\mathbf{i}\zeta)^{\lambda}\kappa(\zeta)$, which is bounded by assumption and therefore absolutely integrable, given the assumed compact support of $\kappa(\zeta)$. Hence $k^{(\lambda)}(z)$ is bounded, since $\left|k^{(\lambda)}(z)\right|=\left|\int(-\mathbf{i}\zeta)^{\lambda}\kappa(\zeta)e^{-i\zeta z}d\zeta\right|\le\int|\zeta|^{\lambda}|\kappa(\zeta)|d\zeta<\infty$. Note that $\int\left|k^{(\lambda)}(z)\right|^2dz>0$ unless $k^{(\lambda)}(z)=0$ for all $z\in\mathbb{R}$, which would imply that $k(z)$ is a polynomial, making it impossible to satisfy $\int k(z)dz=1$. Hence, $\int\left|k^{(\lambda)}(z)\right|^2dz>0$.

The Fourier transform of $z^2k^{(\lambda)}(z)$ is $-(d^2/d\zeta^2)\left((-\mathbf{i}\zeta)^{\lambda}\kappa(\zeta)\right)$. By the compact support of $\kappa(\zeta)$, if $\kappa(\zeta)$ has two bounded derivatives then so does $(-\mathbf{i}\zeta)^{\lambda}\kappa(\zeta)$, and it follows that $-(d^2/d\zeta^2)\left((-\mathbf{i}\zeta)^{\lambda}\kappa(\zeta)\right)$ is absolutely integrable. By the Riemann-Lebesgue Lemma, the inverse Fourier transform of $\mathbf{i}(d^2/d\zeta^2)\left((-\mathbf{i}\zeta)^{\lambda}\kappa(\zeta)\right)$ is such that $z^2k^{(\lambda)}(z)\to 0$ as $|z|\to\infty$. Hence, we know that there exists C such that

$$\left|k^{(\lambda)}(z)\right|\le\frac{C}{1+z^2},$$

and the function on the right-hand side satisfies all the remaining properties stated in the lemma.
■

Proof of Theorem 3.2. (*i*) The order of magnitude of the bias is derived in the proof of Theorem 4.4 in the foregoing appendix. The convergence rate of $B_{V,\lambda}(z,h)$ is also derived in Theorem 4.4.

(*ii*) The facts that $E\left[L_{V,\lambda}(z,h)\right]=0$ and $E\left[L^2_{V,\lambda}(z,h)\right]=n^{-1}\Omega_{V,\lambda}(z,h)$ hold by construction. Next, Assumptions 3.2(*ii*) and 3.4 ensure that

$$\begin{aligned}\Omega_{V,\lambda}(z,h) &= E\left[\left((-1)^{\lambda}h^{-\lambda-1}Vk^{(\lambda)}\left(\frac{Z-z}{h}\right)\right)^2\right]-\left(E\left[(-1)^{\lambda}h^{-\lambda-1}Vk^{(\lambda)}\left(\frac{Z-z}{h}\right)\right]\right)^2\\ &\le E\left[\left((-1)^{\lambda}h^{-\lambda-1}Vk^{(\lambda)}\left(\frac{Z-z}{h}\right)\right)^2\right]\end{aligned}$$

$$
\begin{aligned}
&= h^{-2\lambda-1} E\left[E\left[V^2|Z\right] h^{-1}\left(k^{(\lambda)}\left(\frac{Z-z}{h}\right)\right)^2\right] \\
&\preceq h^{-2\lambda-1} E\left[h^{-1}\left(k^{(\lambda)}\left(\frac{Z-z}{h}\right)\right)^2\right] \\
&\qquad \text{(by Assumption 3.2}(ii)\text{ and Jensen's inequality)} \\
&= h^{-2\lambda-1}\int h^{-1}\left(k^{(\lambda)}\left(\frac{\tilde{z}-z}{h}\right)\right)^2 f_Z(\tilde{z})\,d\tilde{z} \\
&= h^{-2\lambda-1}\int \left(k^{(\lambda)}(u)\right)^2 f_Z(z+hu)\,du \\
&\qquad \text{(after a change of variable from } \tilde{z} \text{ to } z+hu) \\
&\preceq h^{-2\lambda-1}\int \left(k^{(\lambda)}(u)\right)^2 du \qquad \text{(by Assumption 3.1}(i) \\
&\preceq h^{-2\lambda-1} \qquad \text{(by Lemma A.5)}
\end{aligned}
$$

and hence

$$
\sqrt{\sup_{z\in\mathbb{R}} \Omega_{V,\lambda}(z,h)} = O\left(h^{-\lambda-1/2}\right).
$$

We now establish the uniform convergence rate. Using Parseval's identity, we have

$$
\begin{aligned}
L_{V,\lambda}(z,h) &= \hat{E}\left[(-1)^\lambda h^{-\lambda-1} V k^{(\lambda)}\left(\frac{Z-z}{h}\right)\right] - E\left[(-1)^\lambda h^{-\lambda-1} V k^{(\lambda)}\left(\frac{Z-z}{h}\right)\right] \\
&= \frac{1}{2\pi}\int \left(\hat{E}\left[Ve^{\mathbf{i}\zeta Z}\right] - E\left[Ve^{\mathbf{i}\zeta Z}\right]\right)(-\mathbf{i}\zeta)^\lambda \kappa(h\zeta)\, e^{-\mathbf{i}\zeta z} d\zeta,
\end{aligned}
$$

so it follows that

$$
|L_{V,\lambda}(z,h)| \le \frac{1}{2\pi}\int \left|\hat{E}\left[Ve^{\mathbf{i}\zeta Z}\right] - E\left[Ve^{\mathbf{i}\zeta Z}\right]\right| |\zeta|^\lambda |\kappa(h\zeta)|\, d\zeta,
$$

and that

$$
\begin{aligned}
E\left[|L_{V,\lambda}(z,h)|\right] &\le \frac{1}{2\pi}\int E\left[\left|\hat{E}\left[Ve^{\mathbf{i}\zeta Z}\right] - E\left[Ve^{\mathbf{i}\zeta Z}\right]\right|\right] |\zeta|^\lambda |\kappa(h\zeta)|\, d\zeta \\
&\le \frac{1}{2\pi}\int (E[\Big(\hat{E}\left[Ve^{\mathbf{i}\zeta Z}\right] - E\left[Ve^{\mathbf{i}\zeta Z}\right]\Big) \\
&\qquad \times \left(\hat{E}\left[Ve^{\mathbf{i}\zeta Z}\right] - E\left[Ve^{\mathbf{i}\zeta Z}\right]\right)^\dagger])^{1/2} |\zeta|^\lambda |\kappa(h\zeta)|\, d\zeta \\
&\le \frac{1}{2\pi}\int \left(n^{-1} E\left[Ve^{\mathbf{i}\zeta Z} Ve^{-\mathbf{i}\zeta Z}\right]\right)^{1/2} |\zeta|^\lambda |\kappa(h\zeta)|\, d\zeta \\
&= n^{-1/2}\frac{1}{2\pi}\int \left(E\left[V^2\right]\right)^{1/2} |\zeta|^\lambda |\kappa(h\zeta)|\, d\zeta \\
&\preceq n^{-1/2}\int |\zeta|^\lambda |\kappa(h\zeta)|\, d\zeta \\
&= n^{-1/2} h^{-1-\lambda}\int |\xi|^\lambda |\kappa(\xi)|\, d\xi \\
&\preceq n^{-1/2} h^{-\lambda-1}.
\end{aligned}
$$

Hence, by the Markov inequality,

$$\sup_{z\in\mathbb{R}} |L_{V,\lambda}(z,h)| = O_p\left(n^{-1/2}h^{-\lambda-1}\right).$$

When $h_n \to 0$, lemma 1 in the appendix of Pagan and Ullah (1999, p.362) applies to yield:

$$\begin{aligned}
h_n^{2\lambda+1}\Omega_{V,\lambda}(z,h_n) &= E\left[h_n^{-1}\left((-1)^\lambda V k^{(\lambda)}\left(\frac{Z-z}{h_n}\right)\right)^2\right] \\
&\quad - h_n\left(E\left[(-1)^\lambda h_n^{-1} V k^{(\lambda)}\left(\frac{Z-z}{h_n}\right)\right]\right)^2 \\
&= E\left[E\left[V^2|Z\right]h_n^{-1}\left(k^{(\lambda)}\left(\frac{Z-z}{h_n}\right)\right)^2\right] \\
&\quad - h_n\left(E\left[E\left[V|Z\right]h^{-1}k^{(\lambda)}\left(\frac{Z-z}{h_n}\right)\right]\right)^2 \\
&\to E\left[V^2|Z=z\right]f_Z(z)\int\left(k^{(\lambda)}(z)\right)^2 dz.
\end{aligned}$$

By Assumptions 3.1 and 3.2(iii), $E\left[V^2|Z=z\right]f_Z(z) > 0$ for $z \in \mathbb{S}_Z$ and 3.4 ensures $\int\left(k^{(\lambda)}(z)\right)^2 dz > 0$ by Lemma A.5, so that $h_n^{2\lambda+1}\Omega_{V,\lambda}(z,h_n) > 0$ for all n sufficiently large.

(iii) To show asymptotic normality, we verify that $\ell_{V,\lambda}(z,h_n;V,Z)$ satisfies the hypotheses of the Lindeberg-Feller Central Limit Theorem for IID triangular arrays (indexed by n). The Lindeberg condition is: For all $\varepsilon > 0$,

$$\lim_{n\to\infty} Q_{n,h_n}(z,\varepsilon) \to 0,$$

where

$$Q_{n,h}(z,\varepsilon) \equiv \left(\Omega_{V,\lambda}(z,h)\right)^{-1} E\left[1\left(|\ell_{V,\lambda}(z,h;V,Z)| \geq \varepsilon\left(\Omega_{V,\lambda}(z,h)\right)^{1/2}n^{1/2}\right)|\ell_{V,\lambda}(z,h;V,Z)|^2\right].$$

Using the inequality $E\left[1\left[W \geq \eta\right]W^2\right] \leq \eta^{-\delta}E\left[W^{2+\delta}\right]$ for any $\delta > 0$, we have

$$Q_{n,h}(z,\varepsilon) \leq \left(\Omega_{V,\lambda}(z,h)\right)^{-1}\left(\varepsilon\left(\Omega_{V,\lambda}(z,h)\right)^{1/2}n^{1/2}\right)^{-\delta} E\left[|\ell_{V,\lambda}(z,h;V,Z)|^{2+\delta}\right],$$

where Assumption 3.2(iv) ensures that

$$\begin{aligned}
E\left[|\ell_{V,\lambda}(z,h;V,Z)|^{2+\delta}\right] &= h^{-\lambda(2+\delta)}h^{-1-\delta}E\left[h^{-1}|V|^{2+\delta}\left|k^{(\lambda)}\left(\frac{Z-z}{h}\right)\right|^{2+\delta}\right] \\
&= h^{-\lambda(2+\delta)}h^{-1-\delta}E\left[h^{-1}E\left[|V|^{2+\delta}|Z\right]\left|k^{(\lambda)}\left(\frac{Z-z}{h}\right)\right|^{2+\delta}\right] \\
&\preceq h^{-\lambda(2+\delta)}h^{-1-\delta}E\left[h^{-1}\left|k^{(\lambda)}\left(\frac{Z-z}{h}\right)\right|^{2+\delta}\right] \\
&\preceq h^{-\lambda(2+\delta)}h^{-1-\delta}.
\end{aligned}$$

The results above and Assumption 3.2(iv) ensure that for any given z there exist $0 < A_{1,z}, A_{2,z} < \infty$ such that $A_{1,z} h_n^{-2\lambda-1} < \Omega_{V,\lambda}(z, h_n) < A_{2,z} h_n^{-2\lambda-1}$ for all h_n sufficiently small. Hence, we have

$$\begin{aligned} Q_{n,h_n}(z,\varepsilon) &\preceq \left(\varepsilon h_n^{-\lambda-1/2} n^{1/2}\right)^{-\delta} \frac{h_n^{-\lambda(2+\delta)} h_n^{-1-\delta}}{h_n^{-2\lambda-1}} \\ &= \left(\varepsilon h_n^{-\lambda-1/2} n^{1/2} h_n^{\lambda} h_n\right)^{-\delta} \\ &= \varepsilon^{-\delta} (n h_n)^{-\delta/2} \to 0 \end{aligned}$$

provided $nh_n \to \infty$, which is implied by Assumption 3.6: $h_n \to 0, nh_n^{2\lambda+1} \to \infty$. ■

Proof of Theorem 3.3. The $O\left(\left\|\tilde{g}_{V_j,\lambda_j} - g_{V_j,\lambda_j}\right\|_\infty^2\right)$ remainder in eq.(15) can be dealt with as in the proof above of Theorem 4.9. Next, we note that

$$\int s(z)\left(\hat{g}_{V,\lambda}(z,h) - g_{V,\lambda}(z)\right) dz = L + B_h + R_h,$$

where

$$\begin{aligned} L &= \hat{E}\left[V s^{(\lambda)}(Z)\right] - E\left[V s^{(\lambda)}(Z)\right] = \hat{E}\left[\psi_{V,\lambda}(s; V, Z)\right] \\ B_h &= \int s(z)\left(g_{V,\lambda}(z,h) - g_{V,\lambda}(z)\right) dz \\ R_h &= \int s(z)\left(\hat{g}_{V,\lambda}(z,h) - g_{V,\lambda}(z,h)\right) dz - \left(\hat{E}\left[V s^{(\lambda)}(Z)\right] - E\left[V s^{(\lambda)}(Z)\right]\right). \end{aligned}$$

We then have, by Assumption 3.7,

$$\begin{aligned} |B_{h_n}| &\equiv \left|\int s(z)\left(g_{V,\lambda}(z,h_n) - g_{V,\lambda}(z)\right) dz\right| \leq \int |s(z)| \left|g_{V,\lambda}(z,h_n) - g_{V,\lambda}(z)\right| dz \\ &= \int |s(z)| \left|B_{V,\lambda}(z,h_n)\right| dz = o_p\left(n^{-1/2}\right) \int |s(z)|\, dz = o_p\left(n^{-1/2}\right). \end{aligned}$$

Next,

$$\begin{aligned} R_{h_n} &= \int s(z)\left(\hat{g}_{V,\lambda}(z,h_n) - g_{V,\lambda}(z,h_n)\right) dz - \left(\hat{E}\left[s^{(\lambda)}(Z) V\right] - E\left[s^{(\lambda)}(Z) V\right]\right) \\ &= \int s(z)\left(\hat{E}\left[V \frac{1}{h_n^{1+\lambda}} k^{(\lambda)}\left(\frac{Z-z}{h_n}\right)\right] - E\left[V \frac{1}{h_n^{1+\lambda}} k^{(\lambda)}\left(\frac{Z-z}{h_n}\right)\right]\right) dz \\ &\quad - \left(\hat{E}\left[V s^{(\lambda)}(Z)\right] - E\left[V s^{(\lambda)}(Z)\right]\right) \\ &= B + (-1)^\lambda \int s^{(\lambda)}(z)\left(\hat{E}\left[V \frac{1}{h_n} k\left(\frac{Z-z}{h_n}\right)\right] - E\left[V \frac{1}{h_n} k\left(\frac{Z-z}{h_n}\right)\right]\right) dz \\ &\quad - \left(\hat{E}\left[V s^{(\lambda)}(Z)\right] - E\left[V s^{(\lambda)}(Z)\right]\right) \\ &= B + (-1)^\lambda \int (\hat{E}\left[V s^{(\lambda)}(z) \frac{1}{h_n} k\left(\frac{Z-z}{h_n}\right) - V s^{(\lambda)}(Z)\right] \\ &\quad - E\left[V s^{(\lambda)}(z) \frac{1}{h_n} k\left(\frac{Z-z}{h_n}\right) - V s^{(\lambda)}(Z)\right]) dz \\ &= B + (-1)^\lambda \hat{E}\left[V\left(s^{(\lambda)}(Z,h_n) - s^{(\lambda)}(Z)\right) - E\left[V\left(s^{(\lambda)}(Z,h_n) - s^{(\lambda)}(Z)\right)\right]\right] \end{aligned}$$

where

$$s^{(\lambda)}(\tilde{z},h)=\int s^{(\lambda)}(z)\frac{1}{h}k\left(\frac{\tilde{z}-z}{h}\right)dz$$

and where the boundary term B from the integration by parts satisfies

$$|B|\leq\sum_{l=1}^{\lambda}\sum_{p=-1,1}\lim_{pz\to\infty}\left|s^{(\lambda-l)}(z)\right|\left(\left|\hat{E}\left[V\frac{1}{h_n^l}k^{(l-1)}\left(\frac{Z-z}{h_n}\right)\right]\right|+\left|E\left[V\frac{1}{h_n^l}k^{(l-1)}\left(\frac{Z-z}{h_n}\right)\right]\right|\right)=0$$

because at any given h_n, we have $\lim_{|z|\to\infty}\max_{l=1,\ldots\lambda_j-1}\left|s_j^{(l)}(z)\right|=0$ by assumption while the expectations and estimated expectations are bounded since $\left|h_n^{-l}k^{(l-1)}((Z-z)/h_n)\right|$ is bounded (by Lemma A.5) and so is $E[|V||Z=z]$ by Assumption 3.2.

Hence, $R_{h_n}=o_p\left(n^{-1/2}\right)$, because it is a zero-mean sample average where the variance of each individual IID term can be shown to go to zero as $h_n\to 0$ as follows. Lemma A.5 and the assumed uniform continuity of $s^{(\lambda)}(z)$ imply, by Lemma 1 in Pagan & Ullah (1999), that $s^{(\lambda)}(z,h_n)-s^{(\lambda)}(z)\to 0$ uniformly in $z\in\mathbb{R}$ as $h_n\to 0$. Let $\varepsilon_n=\sup_{z\in\mathbb{R}}\left|s^{(\lambda)}(z,h_n)-s^{(\lambda)}(z)\right|\to 0$, we then have

$$\operatorname{Var}\left[V\left(s^{(\lambda)}(Z,h_n)-s^{(\lambda)}(Z)\right)\right]\leq\varepsilon_n^2\operatorname{Var}[V]\to 0.$$

■

Proof of Theorem 3.4. This proof is virtually identical to the proof of Theorem 4.10 in the foregoing appendix, with $\varepsilon_n=\left(h_n^{-1}\right)^{\gamma_{1,B}}\exp\left(\alpha_B\left(h_n^{-1}\right)^{\beta_B}\right)+n^{-1/2}\left(h_n^{-1}\right)^2$ instead of $\varepsilon_n=\left(h_n^{-1}\right)^{\gamma_{1,B}}\exp\left(\alpha_B\left(h_n^{-1}\right)^{\beta_B}\right)+n^{-1/2}\left(h_n^{-1}\right)^{\gamma_{1,L}}\exp\left(\alpha_L\left(h_n^{-1}\right)^{\beta_L}\right)$. ■

Proof of Theorem 3.5. This proof is virtually identical to the proof of Theorem 4.11, invoking Theorem 3.2 instead of Corollary 4.8. ■