Supplementary Material for "Measurement Error in Multiple Equations: Tobin's q and Corporate Investment, Saving, and Debt"

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A Supplementary Material on Identification

A.1 Restricting the Correlations among the Disturbances

We extend A_6 to A'_6 which restricts the sign and/or magnitude of the correlation r_{η_j,η_h} between η_j and η_h.

Assumption $\mathbf{A'_6}$ *Disturbance Correlation Restriction:* $\underline{c}_{jh} \leq r_{\eta_j,\eta_h} \leq \overline{c}_{jh}$ *where* $-1 \leq \underline{c}_{jh} \leq \overline{c}_{jh} \leq 1$ *for* $j, h = 1, ..., p$ *and* $j < h$.

In particular, provided $\sigma^2_{\tilde{Y}_j}\sigma^2_{\tilde{Y}_h} \neq 0$, from the proof of Corollary A.1 we have that

$$r_{\eta_j,\eta_h} = \frac{\rho r_{\tilde{Y}_j,\tilde{Y}_h} - r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h}}{(\rho - R^2_{\tilde{W}.\tilde{Y}_j})^{\frac{1}{2}}(\rho - R^2_{\tilde{W}.\tilde{Y}_h})^{\frac{1}{2}}}.$$

A'_6 may restrict the sign of r_{η_j,η_h} as encoded by the sign of the function

$$S_{jh}(r) \equiv r \times r_{\tilde{Y}_j,\tilde{Y}_h} - r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h}.$$

Further, A'_6 may restrict the magnitude of r_{η_j,η_h} (either $r^2_{\eta_j,\eta_h} \leq c^2$ or $c^2 \leq r^2_{\eta_j,\eta_h}$) as encoded by the sign of the function

$$M_{jh}(r; c) \equiv (r \times r_{\tilde{Y}_j,\tilde{Y}_h} - r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h})^2 - c^2(r - R^2_{\tilde{W}.\tilde{Y}_j})(r - R^2_{\tilde{W}.\tilde{Y}_h}).$$

As shown in the proof of Corollary A.1, when $R^2_{\tilde{Y}_j.\tilde{Y}_h} \neq 1$, the discriminant of the quadratic function $M_{jh}(\cdot; c)$ is given by

$$\Delta_{jh}(c) \equiv c^2[R^4_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})^2 - (1 - c^2)(R^2_{\tilde{W}.\tilde{Y}_j} - R^2_{\tilde{W}.\tilde{Y}_h})^2],$$

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and, when $R^2_{\tilde{Y}_j.\tilde{Y}_h} \neq c^2$, the roots of $M_{jh}(\cdot;c)$ are given by

$$\rho^-_{jh}(c) \equiv \frac{F_{jh}(c) - \Delta_{jh}(c)^{\frac{1}{2}}}{2(R^2_{\tilde{Y}_j.\tilde{Y}_h} - c^2)} \qquad \text{and} \qquad \rho^+_{jh}(c) \equiv \frac{F_{jh}(c) + \Delta_{jh}(c)^{\frac{1}{2}}}{2(R^2_{\tilde{Y}_j.\tilde{Y}_h} - c^2)}$$

where

$$F_{jh}(c) \equiv -R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) + (1-c^2)(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h}).$$

Corollary A.1 uses $S_{jh}(r)$ and $M_{jh}(r;c)$ to encode the sign and magnitudes restrictions in A$'_6$ and to express the identification region for $(\rho, \delta, \beta, \Gamma)$ under A$_1$-A$'_6$.

Corollary A.1 *Under the conditions of Theorem 3.1, A_4, A_5, and A'_6 for $j, h = 1, ..., p$ with $j < h$, $(\rho, \delta, \beta, \Gamma)$ is partially identified in the sharp set*

$$\mathcal{J}^{k,\tau,\mathbf{c}} \equiv \{(r, D(r), B(r), G(r)) : 0 \preceq G(r),\ \frac{1}{1+\kappa} \leq r \leq 1,\ \frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}_j}}(1-\tau_j) \leq G_{jj}(r),$$
$$\text{and } \underline{c}_{jh} \leq \frac{G_{jh}(r)}{[G_{jj}(r)G_{hh}(r)]^{\frac{1}{2}}} \leq \bar{c}_{jh} \text{ for } j, h = 1, ..., p \text{ and } j < h\}.$$

Further, ρ is partially identified in the sharp set

$$\mathcal{R}^{k,\tau,\mathbf{c}} = [R^2_{\tilde{W}.\tilde{Y}}, 1] \cap [\frac{1}{1+\kappa}, 1] \cap_{j=1}^{p} [\frac{1}{\tau_j} R^2_{\tilde{W}.\tilde{Y}_j}, 1] \bigcap_{\substack{j,h=1 \\ j<h}}^{p} \mathcal{R}^{\mathbf{c}}_{jh},$$

with

$$\mathcal{R}^{\mathbf{c}}_{jh} = \left\{ r : \begin{array}{cl} S_{jh}(r) \leq 0 \text{ and } M_{jh}(r;\underline{c}_{jh}) \leq 0 \leq M_{jh}(r;\bar{c}_{jh}) & \text{if } \underline{c}_{jh} \leq \bar{c}_{jh} \leq 0 \text{ and } \sigma^2_{\tilde{Y}_j}\sigma^2_{\tilde{Y}_h} \neq 0 \\ \{S_{jh}(r) \leq 0 \text{ and } M_{jh}(r;\underline{c}_{jh}) \leq 0\} \text{ or } & \\ \{0 \leq S_{jh}(r) \text{ and } M_{jh}(r;\bar{c}_{jh}) \leq 0\} & \text{if } \underline{c}_{jh} < 0 < \bar{c}_{jh} \text{ and } \sigma^2_{\tilde{Y}_j}\sigma^2_{\tilde{Y}_h} \neq 0 \\ 0 \leq S_{jh}(r) \text{ and } M_{jh}(r;\bar{c}_{jh}) \leq 0 \leq M_{jh}(r;\underline{c}_{jh}) & \text{if } 0 \leq \underline{c}_{jh} \leq \bar{c}_{jh} \text{ and } \sigma^2_{\tilde{Y}_j}\sigma^2_{\tilde{Y}_h} \neq 0 \\ r \in \emptyset & \text{if } 0 \notin [\underline{c}_{jh}, \bar{c}_{jh}] \text{ and } \sigma^2_{\tilde{Y}_j}\sigma^2_{\tilde{Y}_h} = 0. \\ -\infty < r < \infty & \text{if } 0 \in [\underline{c}_{jh}, \bar{c}_{jh}] \text{ and } \sigma^2_{\tilde{Y}_j}\sigma^2_{\tilde{Y}_h} = 0. \end{array} \right\},$$

where, provided $\sigma^2_{\tilde{Y}_j}\sigma^2_{\tilde{Y}_h} \neq 0$, we have

$$0 \leq S_{jh}(r) \Leftrightarrow \left\{ \begin{array}{ll} \frac{r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h}}{r_{\tilde{Y}_j,\tilde{Y}_h}} \leq r & \text{when } 0 < r_{\tilde{Y}_j,\tilde{Y}_h} \\ r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h} \leq 0 & \text{when } r_{\tilde{Y}_j,\tilde{Y}_h} = 0 \\ r \leq \frac{r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h}}{r_{\tilde{Y}_j,\tilde{Y}_h}} & \text{when } r_{\tilde{Y}_j,\tilde{Y}_h} < 0 \end{array} \right.,$$

and if $R^2_{\tilde{Y}_j.\tilde{Y}_h} = 1$ *then* $0 \leq M_{jh}(r;c) = (1-c^2)(r - R^2_{\tilde{W}.\tilde{Y}_j})^2$ *whereas if* $R^2_{\tilde{Y}_j.\tilde{Y}_h} \neq 1$ *then*

$$0 \leq M_{jh}(r;c) \Leftrightarrow$$
$$\begin{cases} -\infty < r < \infty & \textit{when } 0 < c^2 < 1 - \frac{R^4_{\tilde{W}.(\tilde{Y}_j.\tilde{Y}_h)'}(1-R^2_{\tilde{Y}_j.\tilde{Y}_h})^2}{(R^2_{\tilde{W}.\tilde{Y}_j} - R^2_{\tilde{W}.\tilde{Y}_h})^2} \\ r \in (-\infty, \rho^-_{jh}(c)] \cup [\rho^+_{jh}(c), \infty) & \textit{when } c^2 = 0 < R^2_{\tilde{Y}_j.\tilde{Y}_h} \textit{ or } 1 - \frac{R^4_{\tilde{W}.(\tilde{Y}_j.\tilde{Y}_h)'}(1-R^2_{\tilde{Y}_j.\tilde{Y}_h})^2}{(R^2_{\tilde{W}.\tilde{Y}_j} - R^2_{\tilde{W}.\tilde{Y}_h})^2} \leq c^2 < R^2_{\tilde{Y}_j.\tilde{Y}_h} \\ r \leq \frac{-R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h}}{R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} - (R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})} & \textit{when } c^2 = R^2_{\tilde{Y}_j.\tilde{Y}_h} \textit{ and } R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} < R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h} \\ 0 \leq (1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} & \textit{when } c^2 = R^2_{\tilde{Y}_j.\tilde{Y}_h} \textit{ and } R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} = R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h} \\ \frac{-R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h}}{R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} - (R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})} \leq r & \textit{when } c^2 = R^2_{\tilde{Y}_j.\tilde{Y}_h} \textit{ and } R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h} < R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} \\ r \in [\rho^+_{jh}(c), \rho^-_{jh}(c)] & \textit{when } R^2_{\tilde{Y}_j.\tilde{Y}_h} < c^2 \end{cases}$$

Last, δ*,* β*, and* Γ *are partially identified in the sharp sets* $\mathcal{D}^{k,\tau,\mathbf{c}} = \{D(r) : r \in \mathcal{R}^{k,\tau,\mathbf{c}}\}$*,* $\mathcal{B}^{k,\tau,\mathbf{c}} = \{B(r) : r \in \mathcal{R}^{k,\tau,\mathbf{c}}\}$*, and* $\mathcal{G}^{k,\tau,\mathbf{c}} = \{G(r) : r \in \mathcal{R}^{k,\tau,\mathbf{c}}\}$*.*

The bounds in Corollary A.1 correspond to those in Corollaries 3.2 and 3.3 when $(\underline{c}_{jh}, \overline{c}_{jh})$ is set to $(-1,0)$, $(0,1)$, $(0,0)$, or $(-1,1)$. In particular, when $R^2_{\tilde{Y}_j.\tilde{Y}_h} \neq 0$, the proof of Corollary A.1 gives

$$\rho^-_{jh}(0) = \rho^+_{jh}(0) = \frac{r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h}}{r_{\tilde{Y}_j,\tilde{Y}_h}} = \frac{\sigma_{\tilde{W},\tilde{Y}_j}\sigma_{\tilde{W},\tilde{Y}_h}}{\sigma^2_{\tilde{W}}\sigma_{\tilde{Y}_j,\tilde{Y}_h}},$$

so that $0 \leq M_{jh}(\rho;0) \Leftrightarrow \rho \in (-\infty,\infty)$ and $M_{jh}(\rho;0) \leq 0 \Leftrightarrow \rho = \frac{\sigma_{\tilde{W},\tilde{Y}_j}\sigma_{\tilde{W},\tilde{Y}_h}}{\sigma^2_{\tilde{W}}\sigma_{\tilde{Y}_j,\tilde{Y}_h}}$. Also,

$$\rho^-_{jh}(-1) = \rho^-_{jh}(1) = R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} \quad \text{and} \quad \rho^+_{jh}(-1) = \rho^+_{jh}(1) = 0.$$

Thus, when $R^2_{\tilde{Y}_j.\tilde{Y}_h} < 1$, $M_{jh}(\rho;1) = M_{jh}(\rho;-1) \leq 0 \Leftrightarrow \rho \in (-\infty,0] \cup [R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}, \infty)$. Since $R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} \leq R^2_{\tilde{W}.\tilde{Y}}$, this magnitude restriction is not binding in $\mathcal{R}^{k,\tau,\mathbf{c}}$. It follows that Corollary A.1 yields the same bound $\mathcal{R}^{k,\tau,\mathbf{c}}$ from Corollary 3.3, with $\mathcal{R}^{\mathbf{c}}_{jh}$ determined by the magnitude restriction encoded in $M_{jh}(\rho;0) \leq 0$ when $\underline{c}_{jh} = \overline{c}_{jh} = 0$ and by the sign restrictions, if any, encoded in $S_{jh}(r)$ otherwise.

A.2 Relaxing the Classical Measurement Error Assumption

Assume A_1 and A_2 and consider removing the classical measurement error assumption A_3 to allow ε to be correlated with U or η:

$$Y' = U\delta + \eta \quad \text{and} \quad W = U + \varepsilon \quad \text{where } Cov(U,\eta) = 0.$$

Here, we dispense with X for simplicity - if $Cov[X,(\varepsilon,\eta)']=0$ then the analysis proceeds analogously after projecting on X. We can express the moments in $Var[(Y',W)']$ by

$$\sigma_W^2=\sigma_U^2+\sigma_\varepsilon^2+2\sigma_{U,\varepsilon}, \qquad \sigma_{W,Y}=\sigma_U^2\delta+\sigma_{U,\varepsilon}\delta+\sigma_{\eta,\varepsilon}, \quad \text{and} \;\; \sigma_Y^2=\delta'\sigma_U^2\delta+\sigma_\eta^2.$$

Let W and U be nondegenerate. Dividing the first equation by $0<\sigma_W^2$ gives

$$1=\rho_u+\rho_\varepsilon+2\frac{\sigma_{U,\varepsilon}}{\sigma_W^2} \qquad \text{where } \rho_u\equiv\frac{\sigma_U^2}{\sigma_W^2} \text{ and } \rho_\varepsilon\equiv\frac{\sigma_\varepsilon^2}{\sigma_W^2}.$$

Further, normalizing the second and third moments by σ_W^2, defining $\zeta\equiv\frac{\sigma_{\eta,\varepsilon}}{\sigma_W^2}$ and $\Gamma\equiv\frac{\sigma_\eta^2}{\sigma_W^2}$, and substituting for $\frac{\sigma_{U,\varepsilon}}{\sigma_W^2}=\frac{1}{2}(1-\rho_u-\rho_\varepsilon)$ gives the nonlinear system of equations

$$b_{Y.W}\equiv\frac{\sigma_{W,Y}}{\sigma_W^2}=\rho_u\delta+\frac{1}{2}(1-\rho_u-\rho_\varepsilon)\delta+\zeta \quad \text{and} \quad \frac{\sigma_Y^2}{\sigma_W^2}=\delta'\rho_u\delta+\Gamma,$$

where the dimension of the unknowns $(\rho_u,\rho_\varepsilon,\zeta,\delta,\Gamma)$ exceeds the number of equations. The system's unknowns must also obey that $Var[(U,\varepsilon,\eta')]$ is positive semi-definite. Nevertheless, without additional assumptions, these restrictions do not identify the elements of δ. This holds even if η and ε are assumed to be uncorrelated so that $\zeta\equiv\frac{\sigma_{\eta,\varepsilon}}{\sigma_W^2}=0$.

In particular, if $\zeta\equiv\frac{\sigma_{\eta,\varepsilon}}{\sigma_W^2}=0$ then (we let $1+\rho_u\neq\rho_\varepsilon$ - otherwise, $b_{Y.W}=\frac{1}{2}(1+\rho_u-\rho_\varepsilon)\delta=0$ does not help identify δ)

$$\delta=D(\rho_u,\rho_\varepsilon)\equiv\frac{2}{1+\rho_u-\rho_\varepsilon}b_{Y.W} \quad \text{and}$$
$$\Gamma=G(\rho_u,\rho_\varepsilon)\equiv\frac{\sigma_Y^2}{\sigma_{\tilde{W}}^2}-\frac{4\rho_u}{(1+\rho_u-\rho_\varepsilon)^2}b'_{Y.W}b_{Y.W}.$$

Further, since $\sigma_{\eta,\varepsilon}=0$, $Var[(U,\varepsilon,\eta')]$ is block-diagonal and therefore $0\preceq Var[(U,\varepsilon,\eta')]$ if and only if $0\preceq Var[(U,\varepsilon)']$ and $0\preceq Var(\eta)$. The first constraint $0\preceq Var[(U,\varepsilon)']$ holds if and only if (where we use $\sigma_{U,\varepsilon}^2\leq\sigma_U^2\sigma_\varepsilon^2$)

$$0\leq\rho_u, \qquad 0\leq\rho_\varepsilon, \qquad \text{and } \frac{1}{4}(1-\rho_u-\rho_\varepsilon)^2\leq\rho_u\rho_\varepsilon.$$

Further, since (recall that we let $0<\rho_u$)

$$0\leq\frac{(1+\rho_u-\rho_\varepsilon)^2}{4\rho_u}=\frac{(1-\rho_u-\rho_\varepsilon)^2}{4\rho_u}+\frac{4(1-\rho_\varepsilon)\rho_u}{4\rho_u}\leq\frac{4\rho_u\rho_\varepsilon}{4\rho_u}+\frac{4(1-\rho_\varepsilon)\rho_u}{4\rho_u}=1,$$

it follows from the proof of Corollary 3.3 that the second constraint $\Gamma\equiv\frac{\sigma_\eta^2}{\sigma_W^2}\succeq 0$ holds if and only if

$$R_{W.Y}^2\leq\frac{(1+\rho_u-\rho_\varepsilon)^2}{4\rho_u}=(\frac{1+\rho_u-\rho_\varepsilon}{2})^2\frac{1}{\rho_u}\leq 1.$$

Clearly, $b_{Y_j.W} = 0$ if and only if $\delta_j = 0$. However, more generally, these constraints fail to bound δ_j. In particular, for any value $d \in \mathbb{R}\backslash\{0\}$ for δ_j there exists a value (r_u, r_ε) for $(\rho_u, \rho_\varepsilon)$ such that $d = D_j(r_u, r_\varepsilon)$ and the constraints on $(\rho_u, \rho_\varepsilon)$ in $0 \preceq Var[(U, \varepsilon, \eta')]$ hold. Specifically, let $r \equiv \frac{1}{d} b_{Y_j.W}$ and set (r_u, r_ε) such that $r^2 \leq r_u \leq \frac{r^2}{R^2_{W.Y}}$ and $r_\varepsilon = 1 + r_u - 2r$. Then $D_j(r_u, r_\varepsilon) \equiv \frac{2}{1+r_u-r_\varepsilon} b_{Y_j.W} = \frac{1}{r} b_{Y_j.W} = d$. Further, the $0 \preceq Var[(U, \varepsilon)']$ constraint holds since $0 < r^2 \leq r_u$ and

$$r_\varepsilon = 1 + r_u - 2r \geq 1 + r^2 - 2r = (1-r)^2 \geq 0, \text{ and}$$
$$r_u r_\varepsilon - \frac{1}{4}(1 - r_u - r_\varepsilon)^2 = r_u(1 + r_u - 2r) - \frac{1}{4}[1 - r_u - (1 + r_u - 2r)]^2 = r_u - r^2 \geq 0.$$

Last, the $\Gamma \equiv \frac{\sigma^2_\eta}{\sigma^2_W} \succeq 0$ constraint holds since $R^2_{W.Y} \leq \frac{r^2}{r_u} = (\frac{1+r_u-r_\varepsilon}{2})^2 \frac{1}{r_u} \leq 1$.

This paper's analysis maintains the classical measurement error assumption A_3. In this case, $\zeta \equiv \frac{\sigma_{\eta,\varepsilon}}{\sigma^2_W} = 0$ and $\frac{\sigma_{U,\varepsilon}}{\sigma^2_W} = 0$ and thus $\rho_\varepsilon = 1 - \rho_u$. This reduces the dimension of the unknown parameters in the equations for $\frac{\sigma_{W,Y}}{\sigma^2_{\tilde{W}}}$ and $\frac{\sigma^2_Y}{\sigma^2_{\tilde{W}}}$ by $p+1$, from $(\rho_u, \rho_\varepsilon, \zeta, \delta, \Gamma)$ to (ρ_u, δ, Γ), and yields two-sided bounds for the elements of δ. It is of interest to derive analytical expressions for the sharp identification regions for ρ_u, ρ_ε, ζ, δ, Γ and β without A_3 under restrictions analogous to A_4-A_6. To keep the scope of the paper manageable, we leave tackling this problem in more detail to other work.

B Supplementary Material on Inference

B.1 Algorithm for Inference on ρ

In order to apply only one algorithm that delivers $\hat{\rho}^l_o(\lambda; 1-\alpha_{21})$, $\hat{\rho}^u_o(\lambda; 1-\alpha_{21})$, and $CI^\rho_{1-\alpha_{21}}(\lambda)$, it is useful to adopt the following notation. For $r \in [0,1]$, we let

$$g^l(\pi; r, \lambda) = (g^l_1(\pi; r, \lambda), ..., g^l_M(\pi; r, \lambda)) \text{ where } g^l_v(\pi; r, \lambda) \equiv r - \rho^l_v(\lambda) \text{ for } v = 1, ..., M, \text{ and}$$
$$g^u(\pi; r, \lambda) = (g^u_1(\pi; r, \lambda), ..., g^u_M(\pi; r, \lambda)) \text{ where } g^u_v(\pi; r, \lambda) \equiv \rho^u_v(\lambda) - r \text{ for } v = 1, ..., M.$$

Thus, $\rho^l_v(\lambda) = -g^l_v(\pi; 0, \lambda)$ and[2] $\rho^u_v(\lambda) = g^u(\pi; 0, \lambda)$. Further, we collect all the lower and upper bounds, denoted by $g^c_v(\pi; r, \lambda)$ for $v = 1, ..., 2M$, into

$$g^c(\pi; r, \lambda) = (g^l(\pi; r, \lambda)', g^u(\pi; r, \lambda)')'.$$

[2]We employ $g^l(\pi; 0, \lambda)$ to transform a lower bound for ρ into an upper bounds for $-\rho$. We then use a single algorithm (for an upper bound) when estimating the lower and upper bounds for ρ.

We estimate $g^c(\pi; r, \lambda)$ using the consistent plug-in estimator $g^c(\hat{\pi}; r, \lambda)$. Using the delta method, the linearly independent subset $g^c_*(\hat{\pi}; r, \lambda)$ of $g^c(\hat{\pi}; r, \lambda)$ (recall that some of bounds in $g^c(\pi; r, \lambda)$ are constant or linearly dependent, e.g. in the single equation case or under the diagonal variance restriction in A_6) is asymptotically normally distributed:

$$\sqrt{n}(g^c_*(\hat{\pi}; r, \lambda) - g^c_*(\pi; r, \lambda)) \xrightarrow{d} N(0, \nabla_\pi g^c_*(\pi; r, \lambda)\Sigma\nabla_\pi g^c_*(\pi; r, \lambda)').$$

Note that $\nabla_\pi g^c(\pi; r, \lambda)$ does not depend on r. Section B.2 collects the expressions for $g^c(\pi; r, \lambda)$, and $\nabla_\pi g^c(\pi; r, \lambda)$.

Next, for each $\ell \in \Lambda_{1-\alpha_{22}}$, we implement algorithm 1 in Chernozhukov, Lee, and Rosen (2013). To compute, $CI^\rho_{1-\alpha_{21}}(\ell)$, we invert a test statistic and perform a grid search over $(0, 1]$. For a thorough discussion of the algorithm[3], we refer the reader to Chernozhukov, Lee, and Rosen (2013) and Chernozhukov, Kim, Lee, and Rosen (2015).

1. Let $\alpha \leq \frac{1}{2}$ and $\mathcal{V}^c \equiv \mathcal{V}^l \cup \mathcal{V}^u \equiv \{1, ..., M\} \cup \{M+1, ..., 2M\}$.

   If the target output is:

   (a) $\hat{\rho}^l_o(\ell; 1-\alpha)$ or $\hat{\rho}^u_o(\ell; 1-\alpha)$ then set $m = l$ or u and $r = 0$.

   (b) $CI^\rho_{1-\alpha}(\ell)$ then set $m = c$ and $r \in (0, 1]$.

2. Set $\tilde{\gamma} = 1 - \frac{0.1}{\log n}$. Simulate S draws $Z_1, .., Z_S$ from $N(0, I_{2M})$.

3. For each $v \in \mathcal{V}^c$, compute[4] $\hat{h}(v; \ell) = [\mathbf{1}(v = 1), ..., \mathbf{1}(v = 2M)][\nabla_\pi g^c(\hat{\pi}; r, \ell)\hat{\Sigma}\nabla_\pi g^c(\hat{\pi}; r, \ell)']^{\frac{1}{2}}$ and set $se(v; \ell) = \frac{1}{\sqrt{n}}\left\|\hat{h}(v; \ell)\right\|$.

4. Define $\mathcal{V}^m_+ = \{v \in \mathcal{V}^m : se(v; \ell) \neq 0\}$. Compute

   $$c_{\mathcal{V}^m}(\tilde{\gamma}; \ell) = \tilde{\gamma}\text{-quantile of } \{\sup_{v \in \mathcal{V}^m_+} \frac{\hat{h}(v; \ell)Z_s}{\left\|\hat{h}(v; \ell)\right\|}, s = 1, ..., S\}$$

   and

   $$\hat{\mathcal{V}}^m = \{v \in \mathcal{V}^m_+ : g^m_v(\hat{\pi}; 0, \ell) \leq \min_{v \in \mathcal{V}^m_+}[g^m_v(\hat{\pi}; 0, \ell) + c_{\mathcal{V}^m}(\tilde{\gamma}; \ell)se(v; \ell)] + 2c_{\mathcal{V}^m}(\tilde{\gamma}; \ell)se(v; \ell)\}.$$

[3] We adjust the algorithm in Chernozhukov, Lee, and Rosen (2013) slightly since some of our bounds are deterministic (e.g $\rho \leq 1$). Specifically, we use the estimated bounds to calculate the critical value. Then we report the smallest upper bound among the precision-corrected estimators and the deterministic bounds.

[4] $\nabla_\pi g^c(\hat{\pi}; r, \ell)\hat{\Sigma}\nabla_\pi g^c(\hat{\pi}; r, \ell)'$ may be positive semi-definite and its matrix square root is computed using a singular value decomposition.

5. Compute

$$c_{\hat{\mathcal{V}}^m}(\ell) = (1-\alpha)\text{-quantile of } \{\sup_{v\in\hat{\mathcal{V}}^m} \frac{\hat{h}(v;\ell)Z_s}{\left\|\hat{h}(v;\ell)\right\|}, s = 1, ..., S\}.$$

6. Compute

$$g_o^m(\hat{\pi}; r, \ell) = \inf_{v\in\mathcal{V}^m} [g_v^m(\hat{\pi}; r, \ell)) + c_{\hat{\mathcal{V}}^m}(\ell) se(v; \ell)]$$

If $m = l$ or u then report

$$\hat{\rho}_o^l(\ell; 1-\alpha) = -g_o^l(\hat{\pi}; 0, \ell) \qquad \text{or} \qquad \hat{\rho}_o^u(\ell; 1-\alpha) = g_o^u(\hat{\pi}; 0, \ell)$$

Otherwise, if $m = c$ then report

$$CI_{1-\alpha}^{\rho}(\ell) = \{r \in (0, 1] : g_o^c(\hat{\pi}; r, \ell) \geq 0\}.$$

In the single equation bounds or when A_6 is not in force, the value ℓ of the nuisance parameters does not affect the bounds. Otherwise, let $t = 1, ..., T$ enumerate the $T \equiv \frac{1}{2}p(p-1)$ (j_t, h_t) pairs, $j_t, h_t = 1, ..., p$ with $j_t < h_t$, that correspond to the first T components of λ. From Corollary 3.2, we have that if ℓ is such that $(\underline{c}_{j_t h_t}, \overline{c}_{j_t h_t}) \neq (-1, 1)$, $sgn(-\ell_{T+t}) \notin [\underline{c}_{j_t h_t}, \overline{c}_{j_t h_t}]$, and $\ell_t = 0$ then $\mathcal{R}_{j_t h_t}^{\mathbf{c}}(\ell) = \emptyset$. As such, we drop these ℓ values from $\Lambda_{1-\alpha_{22}}$ since they have no effect on $CR_{1-\alpha_2}^{\rho} = \bigcup_{\ell\in\Lambda_{1-\alpha_{22}}} CI_{1-\alpha_{21}}^{\rho}(\ell)$. For the remaining values of ℓ in $\Lambda_{1-\alpha_{22}}$, $CI_{1-\alpha_{21}}^{\rho}(\ell)$ depends only on the signs (negative, zero, or positive) of the first T components of $\underset{2T\times 1}{\ell}$. To speed up the computation, we remove from $\Lambda_{1-\alpha_{22}}$ the values that are redundant, so that each admissible sign configuration of the first T components of ℓ is represented only once in $\bigcup_{\ell\in\Lambda_{1-\alpha_{22}}} CI_{1-\alpha_{21}}^{\rho}(\ell)$.

B.2 Delta Method

Recall that the nuisance parameters $\lambda \equiv g^{\lambda}(\pi)$, the vector of lower and upper bounds $g_v^c(\pi; r, \lambda)$ in the intersection bounds algorithm for inference on ρ, and the parameters δ_j, β_{jl}, and Γ_{jh}, $j, h = 1, ..., p$ and $l = 1, ..., k$, (written in the form $\theta \equiv H(\pi; \rho)$) can all be expressed as functions of the vector of estimands

$$\begin{aligned}\underset{1\times B}{\pi'} &\equiv (\underset{1\times p(k+1)}{\pi_1'}, \underset{1\times(p+k)}{\pi_2'}, \underset{1\times p(1+k)}{\pi_3'}, \underset{1\times pk}{\pi_4'}, \underset{1\times k}{\pi_5'}, \underset{1\times p}{\pi_6'}, \underset{1\times\frac{1}{2}p(p-1)}{\pi_7'}) \\ &= [vec(b_{Y.(W,X')'})', b'_{W.(Y,X')'}, (b'_{W.(Y_1,X')'}, ..., b'_{W.(Y_p,X')'}), vec(b_{Y.X})', \\ &\quad b'_{W.X}, \sigma_{\tilde{W}}^{-2}(\sigma_{\tilde{Y}_1}^2, ..., \sigma_{\tilde{Y}_p}^2), \sigma_{\tilde{W}}^{-2}(\sigma_{\tilde{Y}_1,\tilde{Y}_2}, ..., \sigma_{\tilde{Y}_{p-1},\tilde{Y}_p})].\end{aligned}$$

Since the plug-in estimator $\hat{\pi}$ satisfies $\sqrt{n}(\hat{\pi} - \pi) \xrightarrow{d} N(0, \Sigma)$, the delta method gives

$$\begin{aligned}
&\sqrt{n}(\hat{\lambda} - \lambda) \xrightarrow{d} N(0, \nabla_\pi g^\lambda(\pi)\Sigma\nabla_\pi g^\lambda(\pi)'),\\
&\sqrt{n}(g_*^c(\hat{\pi}; r, \lambda) - g_*^c(\pi; r, \lambda)) \xrightarrow{d} N(0, \nabla_\pi g_*^c(\pi; r, \lambda)\Sigma\nabla_\pi g_*^c(\pi; r, \lambda)'), \text{ and}\\
&\sqrt{n}(H(\hat{\pi}; r) - H(\pi; r)) \xrightarrow{d} N(0, \nabla_\pi H(\pi; r)\Sigma\nabla_\pi H(\pi; r)'),
\end{aligned}$$

for any $r \in (0, 1]$. In what follows, we provide expressions for g^λ, $\nabla_\pi g^\lambda(\pi)$, $g^c(\pi; r, \lambda)$, $\nabla_\pi g^c(\pi; r, \lambda)$, $H(\pi; r)$ and $\nabla_\pi H(\pi; r)$.

B.2.1 Nuisance Parameters

The $2T = p(p-1)$ nuisance parameters are collected in

$$\lambda = (\lambda_1, \cdots, \lambda_{2T})' = g^\lambda(\pi) \equiv (\sigma_{\tilde{W}}^{-2}\sigma_{\tilde{Y}_1,\tilde{Y}_2}, ..., \sigma_{\tilde{W}}^{-2}\sigma_{\tilde{Y}_{p-1},\tilde{Y}_p}, b_{\tilde{Y}_1.\tilde{W}}b_{\tilde{Y}_2.\tilde{W}}, ..., b_{\tilde{Y}_{p-1}.\tilde{W}}b_{\tilde{Y}_p.\tilde{W}})'.$$

It follows that, for $t = 1, ..., T$, the components of $\underset{p(p-1)\times B}{\nabla_\pi g^\lambda(\pi)}$ are given by

$$\underset{1\times B}{\nabla_\pi g_t^\lambda(\pi)} = \begin{bmatrix} \underset{1\times[p(k+1)+(p+k)+p(1+k)+pk+k+p]}{0} & \underset{1\times\frac{1}{2}p(p-1)}{\imath_t'} \end{bmatrix},$$

where $\underset{\frac{1}{2}p(p-1)\times 1}{\imath_t}$ is the unit vector with 1 in the t^{th} position and 0 elsewhere, and for $t = T+1, ..., 2T$

$$\underset{1\times B}{\nabla_\pi g_t^\lambda(\pi)} = \begin{bmatrix} \underset{1\times p}{\imath_{j_t}'} \otimes \begin{bmatrix} b_{\tilde{Y}_{h_t}.\tilde{W}} & \underset{1\times k}{0} \end{bmatrix} + \underset{1\times p}{\imath_{h_t}'} \otimes \begin{bmatrix} b_{\tilde{Y}_{j_t}.\tilde{W}} & \underset{1\times k}{0} \end{bmatrix} & \underset{1\times[(p+k)+p(1+k)+pk+k+p+\frac{1}{2}p(p-1)]}{0} \end{bmatrix}.$$

B.2.2 Lower and Upper Intersection bounds

Consider the joint equation bounds with $\lambda = \ell^*$ with $(\underline{c}_{j_t h_t}, \overline{c}_{j_t h_t}) \in \{(-1, 0), (0, 1)\}$ and $sgn(\ell_t^*) \in [\underline{c}_{j_t h_t}, \overline{c}_{j_t h_t}]\backslash\{0\}$ for $t = 1, ..., T$. In this case, we have $g^c(\pi; r, \lambda) = (g^l(\pi; r, \lambda)', g^u(\pi; r, \lambda)')'$ with

$$\underset{M\times 1}{g^l(\pi; r, \ell^*)} = \begin{bmatrix} r - R^2_{\tilde{W}.\tilde{Y}} \\ r - \frac{1}{1+\kappa} \\ r - \frac{1}{\tau_1}R^2_{\tilde{W}.\tilde{Y}_1} \\ \vdots \\ r - \frac{1}{\tau_p}R^2_{\tilde{W}.\tilde{Y}_p} \\ r - \frac{b_{\tilde{Y}_1.\tilde{W}}b_{\tilde{Y}_2.\tilde{W}}}{\sigma_{\tilde{W}}^{-2}\sigma_{\tilde{Y}_1,\tilde{Y}_2}} \\ \vdots \\ r - \frac{b_{\tilde{Y}_{p-1}.\tilde{W}}b_{\tilde{Y}_p.\tilde{W}}}{\sigma_{\tilde{W}}^{-2}\sigma_{\tilde{Y}_{p-1},\tilde{Y}_p}} \end{bmatrix} \quad \text{and} \quad \underset{M\times 1}{g^u(\pi; r, \ell^*)} = \begin{bmatrix} 1-r \\ 1-r \\ 1-r \\ \vdots \\ 1-r \\ \infty \\ \vdots \\ \infty \end{bmatrix}$$

where $M \equiv 2+p+T$ (recall $T \equiv \frac{1}{2}p(p-1)$) and

$$R^2_{\tilde{W}.\tilde{Y}} = b_{\tilde{Y}.\tilde{W}} b_{\tilde{W}.\tilde{Y}} = \sum_{h=1}^{p} b_{Y_h.(W,X')',1} b_{W.(Y',X')',h} \quad \text{and} \quad R^2_{\tilde{W}.\tilde{Y}_j} = b_{\tilde{Y}_j.\tilde{W}} b_{\tilde{W}.\tilde{Y}_j} = b_{Y_j.(W,X')',1} b_{W.(Y_j,X')',1}.$$

The components (for $v = 1, ..., 2M$) of $\underset{2M\times B}{\nabla_\pi g^c(\pi; r, \ell^*)}$ are then given by

1. For $v = 1$

$$\underset{1\times B}{\nabla_\pi g^c_1(\pi; r, \ell^*)} = \begin{bmatrix} \sum_{h=1}^{p} \underset{1\times p}{\imath'_h} \otimes \begin{bmatrix} -b_{W.(Y',X')',h} & \underset{1\times k}{0} \end{bmatrix} & \begin{bmatrix} -\underset{1\times p}{b_{\tilde{Y}.\tilde{W}}} & \underset{1\times k}{0} \end{bmatrix} & 0 \end{bmatrix}$$

2. for $v = 2$,

$$\underset{1\times B}{\nabla_\pi g^c_2(\pi; r, \ell^*)} = 0,$$

3. for $v = 2+j$ and $j = 1, ..., p$

$$\underset{1\times B}{\nabla_\pi g^c_v(\pi; r, \ell^*)} = \begin{bmatrix} \underset{1\times p}{\imath'_j} \otimes \begin{bmatrix} -\frac{1}{\tau_j} b_{\tilde{W}.\tilde{Y}_j} & \underset{1\times k}{0} \end{bmatrix} & \underset{1\times(p+k)}{0} & \underset{1\times p}{\imath'_j} \otimes \begin{bmatrix} -\frac{1}{\tau_j} b_{\tilde{Y}_j.\tilde{W}} & \underset{1\times k}{0} \end{bmatrix} & 0 \end{bmatrix}$$

4. for $v = 2+p+t$ and $t = 1, ..., T$,

$$\underset{1\times B}{\nabla_\pi g^c_v(\pi; r, \ell^*)} = \begin{bmatrix} \underset{1\times p(k+1)}{\nabla_{\pi_1} g^l_v(\pi; r, \ell^*)} & 0 & \underset{1\times \frac{1}{2}p(p-1)}{\nabla_{\pi_7} g^l_v(\pi; r, \ell^*)} \end{bmatrix},$$

where

$$\begin{aligned} \underset{1\times p(k+1)}{\nabla_{\pi_1} g^l_v(\pi; r, \ell^*)} &= \nabla_{\pi_1}\Big(r - \frac{b_{\tilde{Y}_{j_t}.\tilde{W}} b_{\tilde{Y}_{h_t}.\tilde{W}}}{\sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_{j_t},\tilde{Y}_{h_t}}}\Big) \\ &= (\sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_{j_t},\tilde{Y}_{h_t}})^{-1} \{ \underset{1\times p}{\imath'_{j_t}} \otimes \begin{bmatrix} -b_{\tilde{Y}_{h_t}.\tilde{W}} & \underset{1\times k}{0} \end{bmatrix} + \underset{1\times p}{\imath'_{h_t}} \otimes \begin{bmatrix} -b_{\tilde{Y}_{j_t}.\tilde{W}} & \underset{1\times k}{0} \end{bmatrix} \} \end{aligned}$$

and

$$\underset{1\times \frac{1}{2}p(p-1)}{\nabla_{\pi_7} g^l_v(\pi; r, \ell^*)} = \nabla_{\pi_7}\Big(r - \frac{b_{\tilde{Y}_{j_t}.\tilde{W}} b_{\tilde{Y}_{h_t}.\tilde{W}}}{\sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_{j_t},\tilde{Y}_{h_t}}}\Big) = \underset{1\times T}{\imath'_t} \otimes \frac{b_{\tilde{Y}_{j_t}.\tilde{W}} b_{\tilde{Y}_{h_t}.\tilde{W}}}{(\sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_{j_t},\tilde{Y}_{h_t}})^2},$$

5. for $v = 2+p+T+1, ..., 2(2+p+T)$

$$\underset{1\times B}{\nabla_\pi g^c_v(\pi; r, \ell^*)} = 0$$

Above, we set $\lambda = \ell^*$ where $(\underline{c}_{j_t h_t}, \overline{c}_{j_t h_t}) \in \{(-1,0),(0,1)\}$ and $sgn(\ell^*_t) \in [\underline{c}_{j_t h_t}, \overline{c}_{j_t h_t}] \backslash \{0\}$ for $t = 1, ..., T \equiv \frac{1}{2}p(p-1)$. More generally, we consider an arbitrary $\ell \in \Lambda_{1-\alpha_{22}}$ and define the matrix $\underset{2M\times 2M}{P}$ ($M \equiv 2+p+T$) to operationalize how the nuisance parameters λ

determines whether $\mathcal{R}^{\mathbf{c}}_{jh}$ contains an upper bound, lower bound, neither, or both according to Corollary 3.3 (recall that we have already dropped from $\Lambda_{1-\alpha_{22}}$ the values ℓ such that $\mathcal{R}^{\mathbf{c}}_{jh}(\ell) = \emptyset$). In particular, let

$$\underset{2M\times 1}{g^c(\pi; r, \ell)} = P\underset{2M\times 1}{g^c(\pi; r, \ell^*)} \qquad \text{and} \qquad \underset{2M\times B}{\nabla_\pi g^c(\pi; r, \ell)} = P\underset{2M\times B}{\nabla_\pi g^c(\pi; r, \ell^*)}$$

where we set the v^{th} row P_v of P as follows, for $t = 1, ..., \frac{1}{2}p(p-1)$:

1. Set $P = \underset{2M\times 2M}{I}$.
2. If $(\underline{c}_{j_t h_t}, \bar{c}_{j_t h_t}) = (0,0)$ and $\ell_t \neq 0$ then change $P_{M+(2+p+t)}$ to $-\imath_{2+p+t}$.
3. If $(\underline{c}_{j_t h_t}, \bar{c}_{j_t h_t}) \in \{(-1,0),(0,1)\}$ and $sgn(\ell_t) \notin [\underline{c}_{j_t h_t}, \bar{c}_{j_t h_t}]$ then change (a) P_{2+p+t} to $\imath_{M+(2+p+t)}$ and (b) $P_{M+(2+p+t)}$ to $-\imath_{2+p+t}$.
4. If $(\underline{c}_{j_t h_t}, \bar{c}_{j_t h_t}) \in \{(-1,0),(0,1)\}$ and $sgn(\ell_t) \in [\underline{c}_{j_t h_t}, \bar{c}_{j_t h_t}]\backslash\{0\}$ then keep (a) P_{2+p+t} as $\imath_{2+p+t}$ and (b) $P_{M+(2+p+t)}$ as $\imath_{M+(2+p+t)}$.
5. Otherwise, change P_{2+p+t} to $\imath_{M+(2+p+t)}$.

Moreover, for the j^{th} single equation bounds, P mutes the irrelevant bounds as follows:

1. Change P_1 to $\imath_{M+(2+p+1)}$
2. For $h = 1, ..., p$, if $h \neq j$ then change P_{2+h} to $\imath_{M+(2+p+h)}$
3. For $t = 1, ..., \frac{1}{2}p(p-1)$, change P_{2+p+t} to $\imath_{M+(2+p+t)}$.

B.2.3 δ_j, β_{jl}, and Γ_{jh}

Letting π enter explicitly in D_j, we have that, for $j = 1, ..., p$,

$$\delta_j = D_j(\pi; r) \equiv \frac{1}{r} b_{\tilde{Y}_j.\tilde{W}} \qquad \text{and} \qquad \underset{1\times B}{\nabla_\pi D_j(\pi; r)} = \begin{bmatrix} \underset{1\times p}{\imath_j'} \otimes \begin{bmatrix} \frac{1}{r} & \underset{1\times k}{0} \end{bmatrix} & 0 \end{bmatrix}.$$

Similarly, for $j = 1, ..., p$ and $l = 1, ..., k$, we have that

$$\beta_{jl} = B_{jl}(\pi; r) \equiv b_{Y_j.X,l} - b_{W.X,l}\frac{1}{r} b_{\tilde{Y}_j.\tilde{W}} \qquad \text{and}$$

$$\underset{1\times B}{\nabla_\pi B_{jl}(r)} = \begin{bmatrix} \underset{1\times p}{\imath_j'} \otimes \begin{bmatrix} -\frac{1}{r} b_{W.X,l} & \underset{1\times k}{0} \end{bmatrix} & \underset{1\times(p+k)}{0} & \underset{1\times p(k+1)}{0} & \underset{1\times p}{\imath_j'} \otimes \underset{1\times k}{\imath_l'} & -\underset{1\times k}{\imath_l'} \otimes \frac{1}{r} b_{\tilde{Y}_j.\tilde{W}} & \underset{1\times\frac{1}{2}p(p+1)}{0} \end{bmatrix}.$$

Last, for $j, h = 1, ..., p$ and $j \leq h$, Γ_{jh} is given by:

$$\Gamma_{jh} = G_{jh}(\pi; r) \equiv \sigma_{\tilde{W}}^{-2}\sigma_{\tilde{Y}_j,\tilde{Y}_h} - b_{\tilde{Y}_j.\tilde{W}}\frac{1}{r}b_{\tilde{Y}_h.\tilde{W}}.$$

Letting $\underset{1\times\frac{1}{2}p(p+1)}{\imath'_{(j,h)}}$ take the value 1 at the entry (j, h) corresponding to $\sigma_{\tilde{W}}^{-2}\sigma_{\tilde{Y}_j,\tilde{Y}_h}$, we have

$$\underset{1\times B}{\nabla_\pi G_{jh}(\pi; r)} = \left[\begin{array}{cccc} \underset{1\times p}{\imath'_j} \otimes \left[\begin{array}{cc} -\frac{1}{r}b_{\tilde{Y}_h.\tilde{W}} & \underset{1\times k}{0} \end{array} \right] + \underset{1\times p}{\imath'_h} \otimes \left[\begin{array}{cc} -\frac{1}{r}b_{\tilde{Y}_j.\tilde{W}} & \underset{1\times k}{0} \end{array} \right] & 0 & \underset{1\times\frac{1}{2}p(p+1)}{\imath'_{(j,h)}} \end{array} \right].$$

Joint Confidence Regions We sometimes construct a confidence region for the vector of parameters $\beta_{.l} = (\beta_{1l}, ..., \beta_{pl})'$ associated with the variable X_l in the system of Y equations. First, we construct a $1 - \alpha_2$ confidence region $CR^\rho_{1-\alpha_2}$ for $\rho \in \mathcal{R}^{k,\tau,\mathbf{c}}$. For each $r \in CR^\rho_{1-\alpha_2}$, the delta method gives the asymptotic distribution of the plug-in estimator $B_{.l}(\hat{\pi}; r)$ for $B_{.l}(\pi; r) = b_{Y.X,l} - b_{W.X,l}\frac{1}{r}b_{\tilde{Y}.\tilde{W}}$. Specifically, we have that

$$\sqrt{n}(B_{.l}(\hat{\pi}; r) - B_{.l}(\pi; r)) \xrightarrow{d} N(0, \Sigma_{B_{.l}}(r)) \text{ where } \Sigma_{B_{.l}}(r) = \nabla_\pi B_{.l}(\pi; r)\Sigma\nabla_\pi B_{.l}(\pi; r)'$$

and $\underset{p\times B}{\nabla_\pi B_{.l}(\pi; r)}$ stacks the expressions $\underset{1\times B}{\nabla_\pi B_{jl}(\pi; r)}$ for $j = 1, ..., p$ derived above. We construct a $1 - \alpha_1$ confidence region $CR^{B_{.l}}_{1-\alpha_1}(r)$ for $B_{.l}(\pi; r)$, by inverting the following Wald statistic which has an asymptotic χ^2_p distribution:

$$CR^{B_{.l}}_{1-\alpha_1}(r) = \{b_{.l} \in \mathcal{B}^*_{.l} : \sqrt{n}(B_{.l}(\hat{\pi}; r) - b_{.l})'\Sigma^{-1}_{B_{.l}}(r)\sqrt{n}(B_{.l}(\hat{\pi}; r) - b_{.l}) \leq q_{1-\alpha_1}\}.$$

Here, $q_{1-\alpha_1}$ is the $1 - \alpha_1$ quantile of χ^2_p and we perform a grid search over an initial neighborhood $\mathcal{B}^*_{.l}$. For instance, we let $\mathcal{B}^*_{.l}$ be a cube that contains each of the p unidimensional $1 - \alpha_1$ confidence regions (e.g. $c = 1.5 \times c'$ where c' is the $1 - \frac{\alpha_1}{2}$ quantile of a standard normal random variable):

$$\mathcal{B}^*_{.l} = \{(b_{1l}, ..., b_{pl}) : B_{jl}(\hat{\pi}; r) - c*se(B_{jl}(\hat{\pi}; r)) \leq b_{jl} \leq B_{jl}(\hat{\pi}; r) + c*se(B_{jl}(\hat{\pi}; r)) \text{ for } j = 1, ..., p\}.$$

Last, we construct the confidence region $CR^{\beta_{.l}}_{1-\alpha_1-\alpha_2}$ for $\beta_{.l}$ by forming the union:

$$CR^{\beta_{.l}}_{1-\alpha_1-\alpha_2} = \bigcup_{r \in CR^\rho_{1-\alpha_2}} CR^{B_{.l}}_{1-\alpha_1}(r)$$

and use $CR^{\beta_{.l}}_{1-\alpha_1-\alpha_2}$ to form decisions regarding a null hypothesis for $(\beta_{1l}, ..., \beta_{pl})$.

C Extension of the Framework to Panel Data

Consider the unbalanced panel equations with firm fixed effects γ_i:

$$\underset{1\times p}{Y_{it}'} = \underset{1\times p}{\gamma_i'} + \underset{1\times k}{X_{it}'}\underset{k\times p}{\beta} + \underset{1\times 1}{U_{it}}\underset{1\times p}{\delta} + \underset{1\times p}{\eta_{it}'} \quad \text{and} \quad \underset{1\times 1}{W_{it}} = \underset{1\times 1}{U_{it}} + \underset{1\times 1}{\varepsilon_{it}} \quad \text{for } i = 1, ..., n \text{ and } t \in S_i.$$

We assume that the data is missing at random from certain time periods. Specifically, we let T denote[5] the total number of time periods in the panel. For $i = 1, ..., n$, we let S_i denote the subset of T in which the data on firm i are observed, with T_i denoting the cardinality of S_i. When time fixed effects are included, X_{it} contains $T_i - 1$ indicator variables corresponding to the years in S_i. We let $E(\eta_{it}) = \mu_\eta$ and $E(\varepsilon_{it}) = \mu_\varepsilon$ for $i = 1, .., n$ and $t \in S_i$ and we consider the case where n is large relative to T.

Let $\bar{A}_i \equiv \frac{1}{T_i} \sum_{t\in S_i} \underset{a\times 1}{A_{it}}$ and $\ddot{A}_{it} \equiv A_{it} - \bar{A}_i$. The fixed effect γ_i drops out from the $\ddot{Y}_{it}$ equation:

$$\underset{1\times p}{\ddot{Y}_{it}'} = \underset{1\times k}{\ddot{X}_{it}'}\underset{k\times p}{\beta} + \underset{1\times 1}{\ddot{U}_{it}}\underset{1\times p}{\delta} + \underset{1\times p}{\ddot{\eta}_{it}'} \quad \text{and} \quad \underset{1\times 1}{\ddot{W}_{it}} = \underset{1\times 1}{\ddot{U}_{it}} + \underset{1\times 1}{\ddot{\varepsilon}_{it}} \quad \text{for } i = 1, ..., n \text{ and } t \in S_i.$$

Letting $\underset{T_i\times a}{\ddot{A}_i} \equiv [\ddot{A}_{i1}', ..., \ddot{A}_{iT_i}']'$, we obtain the panel analogue of assumption A_1:

$$\underset{T_i\times p}{\ddot{Y}_i} = \underset{T_i\times k}{\ddot{X}_i}\underset{k\times p}{\beta} + \underset{T_i\times 1}{\ddot{U}_i}\underset{1\times p}{\delta} + \underset{T_i\times p}{\ddot{\eta}_i} \quad \text{and} \quad \underset{T_i\times 1}{\ddot{W}_i} = \underset{T_i\times 1}{\ddot{U}_i} + \underset{T_i\times 1}{\ddot{\varepsilon}_i} \quad \text{for } i = 1, ..., n.$$

Suppose that A_2-A_3 hold for this equation. Specifically, let

$$Cov[\eta_{it}, (X_{is}, U_{is})] = 0 \quad \text{and} \quad Cov[\varepsilon_{it}, (X_{is}, U_{is}, \eta_{is})] = 0 \text{ for } i = 1, ..., n \text{ and } t, s \in S_i.$$

This imposes "strict exogeneity" across time periods, as is common when applying a within transformation. Given that $\ddot{A}_{it} \equiv A_{it} - \frac{1}{T_i} \sum_{t\in S_i} A_{it}$, we obtain

$$Cov[\ddot{\eta}_{it}, (\ddot{X}_{is}, \ddot{U}_{is})] = 0 \quad \text{and} \quad Cov[\ddot{\varepsilon}_{it}, (\ddot{X}_{is}, \ddot{U}_{is}, \ddot{\eta}_{is})] = 0 \text{ for } i = 1, ..., n \text{ and } t, s \in S_i.$$

Let the binary indicator I_{it}, for $i = 1, ..., n$ and $t = 1, ..., T$, denote whether the observation (Y_{it}, X_{it}, W_{it}) is missing (at random). Let $\underset{T\times 1}{I_i}$ stack I_{it} for $t = 1, ..., T$. Let

$$\sigma_{\ddot{A}_i, \ddot{B}_i} \equiv E(\underset{a\times T_i}{\ddot{A}_i'}\underset{T_i\times b}{\ddot{B}_i}) = E(\sum_{t\in S_i} \underset{a\times 1}{\ddot{A}_{it}}\underset{1\times b}{\ddot{B}_{it}'}) = \sum_{t=1}^{T} E(I_{it}\underset{a\times 1}{\ddot{A}_{it}}\underset{1\times b}{\ddot{B}_{it}'}) = \sum_{t=1}^{T} E(I_{it})E(\underset{a\times 1}{\ddot{A}_{it}}\underset{1\times b}{\ddot{B}_{it}'}).$$

[5] The number of time periods T should not be confused with the dimension of the nuisance parameter $\underset{2T\times 1}{\lambda}$ in Online Appendix B.

In particular, we have $\sigma_{\ddot{X}_i,\ddot{\eta}_i} = 0$ and $\sigma_{\ddot{X}_i,\ddot{\varepsilon}_i} = 0$. Further, let

$$b_{\ddot{A}_i.\ddot{B}_i} \equiv \sigma^{-2}_{\ddot{B}_i}\sigma_{\ddot{B}_i,\ddot{A}_i} \quad \text{and} \quad \epsilon'_{\ddot{A}_{it}.\ddot{B}_{it}} \equiv \ddot{A}'_{it} - \ddot{B}'_{it} b_{\ddot{A}_i.\ddot{B}_i}.$$

Then, provided $\sigma^2_{\ddot{X}_i}$ is nonsingular,

$$\beta = b_{\ddot{Y}_i.\ddot{X}_i} - b_{\ddot{W}_i.\ddot{X}_i}\delta.$$

Let $\tilde{A}_{it} \equiv \epsilon_{\ddot{A}_{it}.\ddot{X}_{it}}$ and $\tilde{A}_i = [\tilde{A}'_{i1}, ..., \tilde{A}'_{iT_i}]'$. By A_1-A_3, we obtain

$$\underset{T_i\times p}{\tilde{Y}_i} = \underset{T_i\times 1}{\tilde{U}_i}\underset{1\times p}{\delta} + \underset{T_i\times p}{\tilde{\eta}_i} \quad \text{and} \quad \underset{T_i\times 1}{\tilde{W}_i} = \underset{T_i\times 1}{\tilde{U}_i} + \underset{T_i\times 1}{\tilde{\varepsilon}_i}.$$

Further, we have

$$\sigma^2_{\tilde{W}_i} = \sigma^2_{\tilde{U}_i} + \sigma^2_{\tilde{\varepsilon}_i}, \qquad \sigma_{\tilde{W}_i,\tilde{Y}_i} = \sigma_{\tilde{W}_i,\tilde{U}_i}\delta = \sigma^2_{\tilde{U}_i}\delta, \quad \text{and } \sigma^2_{\tilde{Y}_i} = \delta'\sigma^2_{\tilde{U}_i}\delta + \sigma^2_{\tilde{\eta}_i}.$$

Provided $\sigma^2_{\tilde{W}_i}$ is nonsingular, we have

$$b_{\tilde{W}_i.\tilde{Y}_i} = \sigma^{-2}_{\tilde{W}_i}\sigma_{\tilde{W}_i,\tilde{Y}_i} = \rho\delta \quad \text{and} \quad \sigma^{-2}_{\tilde{W}_i}\sigma^2_{\tilde{Y}_i} = \delta'\rho\delta + \sigma^{-2}_{\tilde{W}_i}\sigma^2_{\tilde{\eta}_i},$$

where

$$\rho = \sigma^{-2}_{\tilde{W}_i}\sigma^2_{\tilde{U}_i} = E(\sum_{t\in S_i}\tilde{W}_{it}\tilde{W}'_{it})^{-1}E(\sum_{t\in S_i}\tilde{U}_{it}\tilde{U}'_{it}).$$

Given $\rho \neq 0$, we obtain the representation from Theorem 3.1 and we apply the results of the paper to the transformed variables. For inference, we use the robust standard errors that are clustered at the firm level. For example, we estimate $b_{\ddot{A}_i.\ddot{B}_i}$ and $\underset{T_i\times a}{\epsilon_{\ddot{A}_i.\ddot{B}_i}} = (\epsilon'_{\ddot{A}_{i1}.\ddot{B}_{i1}}, ..., \epsilon'_{\ddot{A}_{iT_i}.\ddot{B}_{iT_i}})'$ using their plug in sample analogues

$$\hat{b}_{\ddot{A}_i.\ddot{B}_i} \equiv (\frac{1}{n}\sum_{i=1}^{n}\underset{b\times T_i}{\ddot{B}'_i}\underset{T_i\times b}{\ddot{B}_i})^{-1}(\frac{1}{n}\sum_{i=1}^{n}\underset{b\times T_i}{\ddot{B}'_i}\underset{T_i\times a}{\ddot{A}_i}) \text{ and } \underset{1\times a}{\hat{\epsilon}'_{\ddot{A}_{it}.\ddot{B}_{it}}} = \ddot{A}'_{it} - \ddot{B}'_{it}\hat{b}_{\ddot{A}_i.\ddot{B}_i}$$

and estimate the asymptotic variance of $\sqrt{n}(\hat{b}_{\ddot{A}_i.\ddot{B}_i} - b_{\ddot{A}_i.\ddot{B}_i})$ by

$$(\frac{1}{n}\sum_{i=1}^{n}\underset{b\times T_i}{\ddot{B}'_i}\underset{T_i\times b}{\ddot{B}_i})^{-1}(\frac{1}{n}\sum_{i=1}^{n}\underset{b\times T_i}{\ddot{B}'_i}\underset{T_i\times a}{\hat{\epsilon}_{\ddot{A}_i.\ddot{B}_i}}\underset{a\times T_i}{\hat{\epsilon}'_{\ddot{A}_i.\ddot{B}_i}}\underset{T_i\times b}{\ddot{B}'_i})(\frac{1}{n}\sum_{i=1}^{n}\underset{b\times T_i}{\ddot{B}'_i}\underset{T_i\times b}{\ddot{B}_i})^{-1}$$

Note that the interpretation of A_4-A_6 applies to the stacked and within-transformed variables. In particular, A_4 assumes that

$$\sigma^2_{\tilde{\varepsilon}_i} = E(\sum_{t\in S_i}\tilde{\varepsilon}^2_{it}) = \sum_{t=1}^{T}E(I_{it})E(\tilde{\varepsilon}^2_{it}) \leq \kappa\sigma^2_{\tilde{U}_i} = \kappa E(\sum_{t\in S_i}\tilde{U}^2_{it}) = \kappa\sum_{t=1}^{T}E(I_{it})E(\tilde{U}^2_{it}).$$

For this to hold, it suffices that $E(\tilde{\varepsilon}_{it}^2) \leq \kappa E(\tilde{U}_{it}^2)$ for $t = 1, ..., T$. A_5 assumes that

$$R^2_{\tilde{Y}_{ji}.\tilde{U}_i} = 1 - \frac{\sigma^2_{\tilde{\eta}_{ji}}}{\sigma^2_{\tilde{Y}_{ji}}} = 1 - \frac{E(\sum_{t\in S_i} \tilde{\eta}^2_{jit})}{E(\sum_{t\in S_i} \tilde{Y}^2_{jit})} = 1 - \frac{\sum_{t=1}^T E(I_{it})E(\tilde{\eta}^2_{jit})}{\sum_{t=1}^T E(I_{it})E(\tilde{Y}^2_{jit})} \leq \tau_j,$$

and it suffices for this that $R^2_{\tilde{Y}_{jit}.\tilde{U}_{it}} = 1 - \frac{\sigma^2_{\tilde{\eta}_{jit}}}{\sigma^2_{\tilde{Y}_{jit}}} \leq \tau_j$ for $t = 1, ..., T$. And A_6 assumes that

$$\underline{c}_{jh} \leq r_{\tilde{\eta}_{ji},\tilde{\eta}_{hi}} = \frac{E(\sum_{t\in S_i} \tilde{\eta}_{jit}\tilde{\eta}_{hit})}{E(\sum_{t\in S_i} \tilde{\eta}^2_{jit})^{\frac{1}{2}} E(\sum_{t\in S_i} \tilde{\eta}^2_{hit})^{\frac{1}{2}}} = \frac{\sum_{t=1}^T E(I_{it})E(\tilde{\eta}_{jit}\tilde{\eta}_{hit})}{[\sum_{t=1}^T E(I_{it})E(\tilde{\eta}^2_{jit})]^{\frac{1}{2}}[\sum_{t=1}^T E(I_{it})E(\tilde{\eta}^2_{hit})]^{\frac{1}{2}}} \leq \bar{c}_{jh},$$

which holds if one imposes the same sign restriction on $Cov(\tilde{\eta}_{jit}, \tilde{\eta}_{hit})$ for $t = 1, ..., T$.

The panel analysis without fixed effects proceeds similarly but omits the within transformation (i.e. it sets $\gamma_i = \gamma$ for $i = 1, ..., n$ and $\ddot{A}_{it} = A_{it} - \frac{1}{\sum_{i=1}^n T_i} \sum_{i=1}^n \sum_{t\in S_i} A_{it}$).

D Mathematical Proofs

Proof of Theorem 3.1: By A_2-A_3, $Cov[(\eta', \varepsilon)', X] = 0$. Since $Var(X)$ is nonsingular, A_1 gives

$$\beta = b_{Y.X} - b_{W.X}\delta.$$

A_2-A_3 also give $\sigma_{\tilde{U},\varepsilon} = 0$ and $\sigma_{\tilde{U},\eta} = \sigma_{\varepsilon,\eta} = 0$. Using $\tilde{\varepsilon} = \varepsilon - E(\varepsilon)$ and $\tilde{\eta} = \eta - E(\eta)$ together with $\tilde{Y}' = \tilde{U}\delta + \tilde{\eta}'$ and $\tilde{W} = \tilde{U} + \tilde{\varepsilon}$, we obtain

$$\sigma^2_{\tilde{W}} = \sigma^2_{\tilde{U}} + \sigma^2_{\varepsilon}, \quad \sigma_{\tilde{W},\tilde{Y}} = \sigma_{\tilde{W},\tilde{U}}\delta = \sigma^2_{\tilde{U}}\delta, \quad \text{and } \sigma^2_{\tilde{Y}} = \delta'\sigma^2_{\tilde{U}}\delta + \sigma^2_{\eta}.$$

Since $Var[(X', U)']$ is nonsingular, $\sigma^2_{\tilde{U}} \neq 0$. Thus, $\sigma^2_{\tilde{W}} \neq 0$ and

$$b_{\tilde{Y}.\tilde{W}} \equiv \sigma^{-2}_{\tilde{W}}\sigma_{\tilde{W},\tilde{Y}} = \rho\delta \quad \text{and} \quad \sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{Y}} = \delta'\rho\delta + \Gamma.$$

Since $\rho \neq 0$, we obtain

$$\begin{aligned}
\delta &= D(\rho) \equiv \frac{1}{\rho} b_{\tilde{Y}.\tilde{W}} \\
\beta &= B(\rho) \equiv b_{Y.X} - b_{W.X}D(\rho) = b_{Y.X} - b_{W.X}\frac{1}{\rho}b_{\tilde{Y}.\tilde{W}}, \text{ and} \\
\Gamma &= G(\rho) \equiv \sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{Y}} - D(\rho)'\rho D(\rho) = \sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{Y}} - b'_{\tilde{Y}.\tilde{W}}\frac{1}{\rho}b_{\tilde{Y}.\tilde{W}}.
\end{aligned}$$

Lemma D.1 *Under the conditions of Theorem 3.1,* $R^2_{\tilde{Y}_j.\tilde{W}} \leq R^2_{\tilde{Y}_j.\tilde{U}}$.

Proof of Lemma D.1: If $\sigma^2_{\tilde{Y}_j} = 0$, set $R^2_{\tilde{Y}_j.\tilde{W}} = R^2_{\tilde{Y}_j.\tilde{U}} = 0$. If $0 < \sigma^2_{\tilde{Y}_j}$, we have

$$R^2_{\tilde{Y}_j.\tilde{W}} = \frac{\sigma^2_{\tilde{W}}}{\sigma^2_{\tilde{Y}_j}} b^2_{\tilde{Y}_j.\tilde{W}} = \frac{\sigma^2_{\tilde{W}}}{\sigma^2_{\tilde{Y}_j}} (\delta_j \rho)^2 \text{ and}$$

$$R^2_{\tilde{Y}_j.\tilde{U}} = 1 - \frac{\sigma^2_{\eta_j}}{\sigma^2_{\tilde{Y}_j}} = \frac{1}{\sigma^2_{\tilde{Y}_j}} (\sigma^2_{\tilde{Y}_j} - \sigma^2_{\eta_j}) = \frac{1}{\sigma^2_{\tilde{Y}_j}} \delta^2_j \sigma^2_{\tilde{U}} = \frac{\sigma^2_{\tilde{W}}}{\sigma^2_{\tilde{Y}_j}} \delta^2_j \rho$$

It follows that

$$R^2_{\tilde{Y}_j.\tilde{U}} - R^2_{\tilde{Y}_j.\tilde{W}} = \frac{\sigma^2_{\tilde{W}}}{\sigma^2_{\tilde{Y}_j}} (\delta^2_j \rho - \delta^2_j \rho^2) = \frac{\sigma^2_{\tilde{W}}}{\sigma^2_{\tilde{Y}}} \rho (1 - \rho) \delta^2_j \geq 0.$$

Proof of Corollary 3.2: The identification set $\mathcal{J}^{k,\tau,\mathbf{c}}$ obtains from A_1-A_6 and the $(Var[(\tilde{Y}', \tilde{W})'])$ moments given by (in)equalities (4-7), using the expressions in Theorem 3.1. To show that $\mathcal{J}^{k,\tau,\mathbf{c}}$ is sharp, let $d = D(r)$, $b = B(r)$, and $g = G(r)$. We show that for each $(r, d, b, g) \in \mathcal{J}^{k,\tau,\mathbf{c}}$ there exist random variables $(U^*, \eta^{*\prime}, \varepsilon^*)$ such that $Y' = X'b + U^* d + \eta^{*\prime}$ and $W = U^* + \varepsilon^*$ that satisfy A_2-A_6. Specifically, $(X, U^*, \varepsilon^*, \eta^{*\prime})$ satisfy A_2-A_3, $Cov[\eta^*, (X', U^*)'] = 0$, $Cov[\varepsilon^*, (\eta^{*\prime}, X', U^*)'] = 0$. Further, $\frac{\sigma^2_{\tilde{U}^*}}{\sigma^2_{\tilde{W}}} = r$ and thus A_4 holds, $\sigma^2_{\varepsilon^*} \leq \kappa \sigma^2_{\tilde{U}^*}$. Last, $G(r) = \sigma^{-2}_{\tilde{W}} \sigma^2_{\tilde{\eta}^*}$ and therefore A_5 holds since, when $\sigma^2_{\tilde{Y}_j} \neq 0$,

$$1 - \frac{\sigma^2_{\eta^*_j}}{\sigma^2_{\tilde{Y}_j}} = \frac{\sigma^2_{\tilde{W}}}{\sigma^2_{\tilde{Y}_j}} \left(\frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}}} - G_{jj}(r) \right) \leq \frac{\sigma^2_{\tilde{W}}}{\sigma^2_{\tilde{Y}_j}} \left[\frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}}} - \frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}}} (1 - \tau_j) \right] = \tau_j,$$

and A_6 holds since $\underline{c}_{jh} \leq sgn(G_{jh}(r)) \leq \bar{c}_{jh}$.

To construct these variables we proceed similarly to Chalak and Kim (2019, proof of corollary 3.2). In particular, we let V be any random variable such that $\tilde{V} \equiv \epsilon_{V.X}$ is nondegenerate and satisfies

$$\sigma_{\tilde{W}.\tilde{V}} = \sqrt{r} \sigma_{\tilde{V}} \sigma_{\tilde{W}} \quad \text{and} \quad \sigma_{\tilde{Y}.\tilde{V}} = \frac{1}{\sqrt{r}} \sigma_{\tilde{V}} \sigma_{\tilde{W}} \frac{\sigma_{\tilde{Y}.\tilde{W}}}{\sigma^2_{\tilde{W}}}.$$

Note that these covariance restrictions are coherent. Specifically,

$$Var(\tilde{V}, \tilde{W}, \tilde{Y}') = \begin{bmatrix} \sigma^2_{\tilde{V}} & \sqrt{r} \sigma_{\tilde{V}} \sigma_{\tilde{W}} & \frac{\sigma_{\tilde{V}} \sigma_{\tilde{W}}}{\sqrt{r}} \frac{\sigma_{\tilde{W}.\tilde{Y}}}{\sigma^2_{\tilde{W}}} \\ \sqrt{r} \sigma_{\tilde{V}} \sigma_{\tilde{W}} & \sigma^2_{\tilde{W}} & \sigma_{\tilde{W}.\tilde{Y}} \\ \frac{\sigma_{\tilde{V}} \sigma_{\tilde{W}}}{\sqrt{r}} \frac{\sigma_{\tilde{Y}.\tilde{W}}}{\sigma^2_{\tilde{W}}} & \sigma_{\tilde{Y},\tilde{W}} & \sigma^2_{\tilde{Y}} \end{bmatrix}$$

is positive semi-definite because $0 < \sigma^2_{\tilde{V}}$ and its Schur complement

$$0 \preceq \sigma^2_{(\tilde{W}, \tilde{Y}')'} - \sigma_{(\tilde{W}, \tilde{Y}')', \tilde{V}} \sigma^{-2}_{\tilde{V}} \sigma_{\tilde{V}, (\tilde{W}, \tilde{Y}')'} = \begin{bmatrix} (1 - r) \sigma^2_{\tilde{W}} & 0 \\ 0 & \sigma^2_{\tilde{W}} G(r) \end{bmatrix}$$

is positive semi-definite since it is block diagonal with $0 \leq (1-r)\sigma^2_{\tilde{W}}$ and $0 \preceq G(r)$.

For instance, to construct V, set $\sigma_{\tilde{V}}$ to some value (e.g. $\sigma_{\tilde{V}} = 1$) and let ϑ be any random variable that is uncorrelated with $(X', W, Y')'$ (e.g. a residual from a regression on (X', W, Y')). When $\sigma^2_{(\tilde{W},\tilde{Y}')'}$ is nonsingular, one can use the above restrictions on $\sigma_{\tilde{W}.\tilde{V}}$ and $\sigma_{\tilde{Y}.\tilde{V}}$ to construct $b_{\tilde{V}.(\tilde{W},\tilde{Y}')'}$ and the scalar

$$\varkappa = \{\frac{1}{\sigma^2_{\vartheta}}[\sigma^2_{\tilde{V}} - b'_{\tilde{V}.(\tilde{W},\tilde{Y}')'}\sigma^2_{(\tilde{W},\tilde{Y}')}b_{\tilde{V}.(\tilde{W},\tilde{Y}')'}]\}^{\frac{1}{2}}$$

($\varkappa$ is set such that the variance of the generated $\tilde{V}$ is $\sigma^2_{\tilde{V}}$) in order to generate

$$\tilde{V} = (\tilde{W}, \tilde{Y})b_{\tilde{V}.(\tilde{W},\tilde{Y}')'} + \varkappa\vartheta.$$

If $\sigma^2_{(\tilde{W},\tilde{Y})'}$ is singular, one can generate $\tilde{V}$ by omitting the redundant $\tilde{Y}$ components from the above regression construction. Last, $V = X'b_{V.X} + \tilde{V} + E[V - X'b_{V.X}]$ obtains by setting $b_{V.X}$ and $E(V)$ to some value (e.g. zero).

Then it suffices to construct U^*, ε^*, and η^* as follows

$$W \equiv (X', V)b_{W.(X',V)'} + \{\epsilon_{W.(X',V)'} + E[W - (X', V)b_{W.(X',V)'}]\} \equiv U^* + \varepsilon^*,$$

and, if $r \neq 1$,

$$Y \equiv (X', V, \varepsilon^*)b_{Y.(X',V,\varepsilon^*)'} + \{\epsilon_{Y.(X',V,\varepsilon^*)'} + E[Y - (X', V, \varepsilon^*)b_{Y.(X',V,\varepsilon^*)'}]\} \equiv (X', V, \varepsilon^*)b_{Y.(X',V,\varepsilon^*)'} + \eta^*$$

whereas if $r = 1$ then $r_{\tilde{W},\tilde{V}} = 1$ and $\epsilon_{W.(X',V)'} = \epsilon_{\tilde{W}.\tilde{V}} = 0$ and

$$Y = (X', V)b_{Y.(X',V)'} + \{\epsilon_{Y.(X',V)'} + E[Y - (X', V)b_{Y.(X',V)'}]\} \equiv (X', V)b_{Y.(X',V)'} + \eta^*.$$

In particular, $(X, U^*, \varepsilon^*, \eta^*)$ satisfy A_2-A_3 since by construction $Cov[\eta^*, (X', U^*)'] = 0$ and $Cov[\varepsilon^*, (\eta^*, X', U^*)'] = 0$. To verify that A_1 holds, note that if $r \neq 1$,

$$\begin{aligned}
Y &= Vb_{\tilde{Y}.\tilde{V}} + X'(b_{Y.X} - b_{V.X}b_{\tilde{Y}.\tilde{V}}) + \varepsilon^* b_{Y.\varepsilon^*} + \{\epsilon_{Y.(X',V,\varepsilon^*)'} + E[Y - (X', V, \varepsilon^*)b_{Y.(X',V,\varepsilon^*)'}]\} \\
&= Vb_{\tilde{W}.\tilde{V}}d + X'(b_{W.X} - b_{V.X}b_{\tilde{W}.\tilde{V}})d + X'(b_{Y.X} - b_{W.X}d) + \varepsilon^* b_{Y.\varepsilon^*} + \eta^* \\
&= (X', V)b_{W.(X',V)'}d + X'b + \eta^* \\
&\equiv U^*d + X'b + \eta^*
\end{aligned}$$

where the first equality uses $Cov[\varepsilon^*, (X', V)'] = 0$ and partitioned regression, the second equality makes use of

$$b_{\tilde{Y}.\tilde{V}} = \sigma^{-2}_{\tilde{W}}\sigma_{\tilde{Y}.\tilde{V}} = \sigma^{-2}_{\tilde{W}}\frac{1}{\sqrt{r}}\sigma_{\tilde{W}}\sigma_{\tilde{V}}\frac{\sigma_{\tilde{Y}.\tilde{W}}}{\sigma^2_{\tilde{W}}} = \sigma^{-2}_{\tilde{W}}\frac{1}{r}\sigma_{\tilde{W}.\tilde{V}}b_{\tilde{Y}.\tilde{W}} = b_{\tilde{W}.\tilde{V}}d,$$

and the third equality uses partitioned regression, $b = b_{Y.X} - b_{W.X}d$, and

$$b_{Y.\varepsilon^*} = b_{\tilde{Y}.\epsilon_{W.(X',V)'}} = \frac{\sigma_{\tilde{Y},\epsilon_{W.(X',V)'}}}{\sigma^2_{\epsilon_{W.(X',V)'}}} = \frac{1}{\sigma^2_{\epsilon_{\tilde{W}.\tilde{V}}}} Cov(\tilde{Y}, \tilde{W} - \tilde{V} b_{\tilde{W}.\tilde{V}})$$
$$= \frac{1}{(1-r)\sigma^2_{\tilde{W}}}[\sigma_{\tilde{Y}.\tilde{W}} - \frac{(\frac{1}{\sqrt{r}}\sigma_{\tilde{W}}\sigma_{\tilde{V}}\frac{\sigma_{\tilde{Y},\tilde{W}}}{\sigma^2_{\tilde{W}}})\sqrt{r}\sigma_{\tilde{V}}\sigma_{\tilde{W}}}{\sigma^2_{\tilde{V}}}] = 0.$$

If $r = 1$, a similar calculation gives,

$$Y = (X', V)b_{Y.(X',V)'} + \{\epsilon_{Y.(X',V)'} + E[Y - (X', V)b_{Y.(X',V)'}]\}$$
$$= (X', V)b_{W.(X',V)'}d + X'b + \eta^* \equiv U^*d + X'b + \eta^*.$$

Last, to verify that A_4-A_6 hold, it suffices to verify that

$$\frac{\sigma^2_{\tilde{U}^*}}{\sigma^2_{\tilde{W}}} = \frac{Var(\tilde{V} b_{\tilde{W}.\tilde{V}})}{\sigma^2_{\tilde{W}}} = \frac{\sigma^2_{\tilde{W}.\tilde{V}}}{\sigma^2_{\tilde{V}}\sigma^2_{\tilde{W}}} = r, \text{ and}$$
$$G(r) = \sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{Y}} - b'_{\tilde{Y}.\tilde{W}}\frac{1}{r}b_{\tilde{Y}.\tilde{W}} = \sigma^{-2}_{\tilde{W}}(d'\sigma^2_{\tilde{U}^*}d + \sigma^2_{\tilde{\eta}^*}) - b'_{\tilde{Y}.\tilde{W}}\frac{1}{r}b_{\tilde{Y}.\tilde{W}} = \sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{\eta}^*}.$$

Proof of Corollary 3.3: We start by deriving the identification region $\mathcal{R}^{k,\tau,\mathbf{c}}$ for ρ. First, we show that $R^2_{\tilde{W}.\tilde{Y}} \leq \rho \leq 1$. If $\sigma_{\tilde{Y},\tilde{W}} = 0$ or $\sigma^2_{\tilde{Y}} = 0$ then set $R^2_{\tilde{W}.\tilde{Y}} = 0 \leq \rho \leq 1$. Suppose that $\sigma_{\tilde{Y},\tilde{W}} \neq 0$. Since $0 < \rho$ and $0 \preceq \Gamma$ then for any vector $\underset{p\times 1}{x}$, we have

$$0 \leq \rho x'\sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{Y}}x - x'b'_{\tilde{Y}.\tilde{W}}b_{\tilde{Y}.\tilde{W}}x.$$

Suppose that $\sigma^2_{\tilde{Y}}$ is positive definite so that $0 < \sigma_{\tilde{W},\tilde{Y}}\sigma^{-2}_{\tilde{Y}}\sigma_{\tilde{Y},\tilde{W}}$ (this is without loss of generality since we can drop the redundant $\tilde{Y}$ components otherwise). In particular, for $x = \sigma^{-2}_{\tilde{Y}}\sigma_{\tilde{Y},\tilde{W}}$, we obtain

$$R^2_{\tilde{W}.\tilde{Y}} = \sigma^{-2}_{\tilde{W}}\sigma_{\tilde{W},\tilde{Y}}\sigma^{-2}_{\tilde{Y}}\sigma_{\tilde{Y},\tilde{W}} = \frac{(\sigma_{\tilde{W},\tilde{Y}}\sigma^{-2}_{\tilde{Y}})\sigma_{\tilde{Y},\tilde{W}}\sigma^{-2}_{\tilde{W}}\sigma^{-2}_{\tilde{W}}\sigma_{\tilde{W},\tilde{Y}}(\sigma^{-2}_{\tilde{Y}}\sigma_{\tilde{Y},\tilde{W}})}{(\sigma_{\tilde{W},\tilde{Y}}\sigma^{-2}_{\tilde{Y}})\sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{Y}}(\sigma^{-2}_{\tilde{Y}}\sigma_{\tilde{Y},\tilde{W}})} \leq \rho \leq 1.$$

Second, by A_4, we have $1 - \rho = \frac{\sigma^2_{\tilde{\varepsilon}}}{\sigma^2_{\tilde{W}}} \leq \kappa\frac{\sigma^2_{\tilde{U}}}{\sigma^2_{\tilde{W}}} = \kappa\rho$ and thus $\rho \in [\frac{1}{1+\kappa}, 1]$. Third, by A_5, we have that for $j = 1, ..., p$, $R^2_{\tilde{Y}_j.\tilde{U}} = (1 - \frac{\sigma^2_{\tilde{\eta}_j}}{\sigma^2_{\tilde{Y}_j}}) \leq \tau_j$ (recall that if $\sigma^2_{\tilde{Y}_j} = 0$ then we set $R^2_{\tilde{Y}_j.\tilde{U}} = R^2_{\tilde{W}.\tilde{Y}_j} = 0$). Multiplying by $\frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}}}$ and substituting for Γ_{jj} we obtain

$$b'_{\tilde{Y}_j.\tilde{W}}\frac{1}{\rho}b_{\tilde{Y}_j.\tilde{W}} = \frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}}} - (\sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{Y}_j} - b'_{\tilde{Y}_j.\tilde{W}}\frac{1}{\rho}b_{\tilde{Y}_j.\tilde{W}}) \leq \tau_j\frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}}}$$

and thus $\frac{1}{\tau_j} R^2_{\tilde{W}.\tilde{Y}_j} = \frac{1}{\tau_j} b^2_{\tilde{Y}_j.\tilde{W}} \frac{\sigma^2_{\tilde{W}}}{\sigma^2_{\tilde{Y}_j}} \leq \rho \leq 1$. Last, $\mathcal{R}^{\mathbf{c}}_{jh}$ obtains since $0 < \rho$ and $\Gamma_{jh} = G_{jh}(\rho) = \sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_j,\tilde{Y}_h} - b_{\tilde{Y}_j.\tilde{W}} \frac{1}{\rho} b_{\tilde{Y}_h.\tilde{W}}$ so that

$$G_{jh}(\rho) \leq 0 \text{ if and only if } \begin{cases} \frac{b_{\tilde{Y}_j.\tilde{W}} b_{\tilde{Y}_h.\tilde{W}}}{\sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_j.\tilde{Y}_h}} \leq \rho & \text{when } \sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_j.\tilde{Y}_h} < 0 \\ 0 \leq b_{\tilde{Y}_j.\tilde{W}} b_{\tilde{Y}_h.\tilde{W}} & \text{when } \sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_j.\tilde{Y}_h} = 0 \\ \rho \leq \frac{b_{\tilde{Y}_j.\tilde{W}} b_{\tilde{Y}_h.\tilde{W}}}{\sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_j.\tilde{Y}_h}} & \text{when } 0 < \sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_j.\tilde{Y}_h} \end{cases}.$$

Combining the results, we have $\rho \in \mathcal{R}^{k,\tau,\mathbf{c}} = [R^2_{\tilde{W}.\tilde{Y}}, 1] \cap [\frac{1}{1+\kappa}, 1] \cap_{j=1}^{p} [\frac{1}{\tau_j} R^2_{\tilde{W}.\tilde{Y}_j}, 1] \bigcap_{\substack{j,h=1 \\ j<h}}^{p} \mathcal{R}^{\mathbf{c}}_{jh}$.

To show that $\mathcal{R}^{k,\tau,\mathbf{c}}$ is sharp, it suffices to show that every $r \in \mathcal{R}^{k,\tau,\mathbf{c}}$ corresponds to a point $(r, d, b, g) \in \mathcal{J}^{k,\tau,\mathbf{c}}$. Let $r \in \mathcal{R}^{k,\tau,\mathbf{c}}$. First, we show that $0 \preceq G(r)$. If $R^2_{\tilde{W}.\tilde{Y}} = 0$ then $G(r) = \sigma^{-2}_{\tilde{W}} \sigma^2_{\tilde{Y}} \succeq 0$. Otherwise, note that

$$G(1) = \sigma^{-2}_{\tilde{W}} \sigma^2_{\tilde{Y}} - b'_{\tilde{Y}.\tilde{W}} b_{\tilde{Y}.\tilde{W}} = \sigma^{-2}_{\tilde{W}} [\sigma^2_{\tilde{Y}} - \sigma_{\tilde{Y},\tilde{W}} \sigma^{-2}_{\tilde{W}} \sigma_{\tilde{W},\tilde{Y}}] = \sigma^{-2}_{\tilde{W}} E(\epsilon_{\tilde{Y}.\tilde{W}} \epsilon'_{\tilde{Y}.\tilde{W}}) \succeq 0.$$

Further, when $R^2_{\tilde{W}.\tilde{Y}} \neq 0$, $0 \preceq G(R^2_{\tilde{W}.\tilde{Y}})$. Specifically, $0 < \sigma^4_{\tilde{W}} R^2_{\tilde{W}.\tilde{Y}}$ and

$$\sigma^4_{\tilde{W}} R^2_{\tilde{W}.\tilde{Y}} G(R^2_{\tilde{W}.\tilde{Y}}) = (R^2_{\tilde{W}.\tilde{Y}} \sigma^2_{\tilde{W}}) \sigma^2_{\tilde{Y}} - \sigma_{\tilde{Y},\tilde{W}} \sigma_{\tilde{W},\tilde{Y}} = Var(b'_{\tilde{W}.\tilde{Y}} \tilde{Y}) \sigma^2_{\tilde{Y}} - \sigma_{\tilde{Y},\tilde{W}} \sigma_{\tilde{W},\tilde{Y}} \succeq 0$$

since and for any vector $\underset{p\times 1}{x}$, applying the Cauchy–Schwarz inequality gives

$$\begin{aligned} & x' Var(b'_{\tilde{W}.\tilde{Y}} \tilde{Y}) \sigma^2_{\tilde{Y}} x - x' \sigma_{\tilde{Y},\tilde{W}} \sigma_{\tilde{W},\tilde{Y}} x \\ & = Var(b'_{\tilde{W}.\tilde{Y}} \tilde{Y}) Var(x' \tilde{Y}) - [Cov(x' \tilde{Y}, \tilde{W})]^2 \\ & = Var(b'_{\tilde{W}.\tilde{Y}} \tilde{Y}) Var(x' \tilde{Y}) - [Cov(x' \tilde{Y}, b'_{\tilde{W}.\tilde{Y}} \tilde{Y})]^2 \geq 0 \end{aligned}$$

where we make use of $\tilde{W}' = \tilde{Y}' b_{\tilde{W}.\tilde{Y}} + \epsilon'_{\tilde{W}.\tilde{Y}}$ and $Cov(\tilde{Y}, \epsilon_{\tilde{W}.\tilde{Y}}) = 0$ in the last equality. Since $r \in \mathcal{R}^{k,\tau,\mathbf{c}} \subseteq [R^2_{\tilde{W}.\tilde{Y}}, 1]$, there exists $0 \leq \lambda \leq 1$ such that $\frac{1}{r} = \lambda + (1-\lambda) \frac{1}{R^2_{\tilde{W}.\tilde{Y}}}$ and it follows that

$$0 \preceq G(r) = \lambda G(1) + (1-\lambda) G(R^2_{\tilde{W}.\tilde{Y}}).$$

Clearly, $\frac{1}{1+\kappa} \leq r \leq 1$. Further, for $j = 1, ..., p$, if $\sigma^2_{\tilde{Y}_j} \neq 0$ then $\frac{1}{\tau_j} b^2_{\tilde{Y}_j.\tilde{W}} \frac{\sigma^2_{\tilde{W}}}{\sigma^2_{\tilde{Y}_j}} = \frac{1}{\tau_j} R^2_{\tilde{W}.\tilde{Y}_j} \leq r$ implies that $\frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}}}(1-\tau_j) \leq \frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}}} - b^2_{\tilde{Y}_j.\tilde{W}} \frac{1}{r} = G_{jj}(r)$ (if $\sigma^2_{\tilde{Y}_j} = 0$ then $0 = \frac{\sigma^2_{\tilde{Y}_j}}{\sigma^2_{\tilde{W}}}(1-\tau_j) \leq G_{jj}(r) = 0$). Last, from the expression for $G_{jh}(r) = \sigma^{-2}_{\tilde{W}} \sigma_{\tilde{Y}_j,\tilde{Y}_h} - b_{\tilde{Y}_j.\tilde{W}} \frac{1}{r} b_{\tilde{Y}_h.\tilde{W}}$, we have that $\underline{c}_{jh} \leq sgn(G_{jh}(r)) \leq \bar{c}_{jh}$ for $j, h = 1, ..., p$ with $j < h$.

The sharp bounds $\mathcal{D}^{k,\tau}$, $\mathcal{B}^{k,\tau}$, and $\mathcal{G}^{k,\tau}$ for δ, β, and Γ follow from the mappings $D(\cdot)$, $B(\cdot)$, and $G(\cdot)$ in Theorem 3.1.

Proof of Theorem 5.1: First, for random column vectors A and B, we collect the regression intercept and slope estimands as follows

$$A' = [E(A)' - E(B)'b_{A.B}] + B'b_{A.B} + \epsilon'_{A.B} \equiv (1, B')b^*_{A.B} + \epsilon'_{A.B}.$$

Given observations $\{A_i, B_i\}_{i=1}^n$, denote the linear regression intercept ($\hat{b}^0_{A.B}$) and slope ($\hat{b}_{A.B}$) estimators and the sample residual ($\hat{\epsilon}_{A.B,i}$) by:

$$\tilde{b}_{A.B} = (\hat{b}^0_{A.B}, \hat{b}'_{A.B})' \equiv (\frac{1}{n}\sum_{i=1}^n (1, B'_i)'(1, B'_i))^{-1}(\frac{1}{n}\sum_{i=1}^n (1, B'_i)'A'_i) \text{ and } \hat{\epsilon}'_{A.B,i} \equiv A'_i - (1, B'_i)\tilde{b}_{A.B}.$$

Further, we collect into π^* the following estimands

$$\pi^* \equiv [vec(b^*_{Y.(W,X')'})', b^{*\prime}_{W.(Y',X')'}, b^{*\prime}_{W.(Y_1,X')'}, ..., b^{*\prime}_{W.(Y_p,X')'}, vec(b^*_{Y.X})', b^{*\prime}_{W.X}, \sigma^{-2}_{\tilde{W}} vec(\sigma^2_{\tilde{Y}})']',$$

and into $\tilde{\pi}$ the corresponding estimators:

$$\tilde{\pi} \equiv [vec(\tilde{b}_{Y.(W,X')'})', \tilde{b}'_{W.(Y',X')'}, \tilde{b}'_{W.(Y_1,X')'}, ..., \tilde{b}'_{W.(Y_p,X')'}, vec(\tilde{b}_{Y.X})', \tilde{b}'_{W.X}, \hat{\sigma}^{-2}_{\tilde{W}} vec(\hat{\sigma}^2_{\tilde{Y}})']'.$$

Last, let $\hat{\mu}^2_A = \frac{1}{n}\sum_{i=1}^n A_i A'_i$,

$$\hat{Q} \equiv diag\{\underset{p\times p}{I} \otimes \hat{\mu}^2_{(1,W,X')'}, \hat{\mu}^2_{(1,Y',X')'}, \hat{\mu}^2_{(1,Y_1,X')'}, ..., \hat{\mu}^2_{(1,Y_p,X')'}, \underset{p\times p}{I} \otimes \hat{\mu}^2_{(1,X')'}, \hat{\mu}^2_{(1,X')'}, \underset{\frac{1}{2}p(p+1)\times\frac{1}{2}p(p+1)}{I} \otimes \hat{\sigma}^2_{\tilde{W}}\}.$$

and

$$L \equiv \frac{1}{n}\sum_{i=1}^n [vec((1, W_i, X'_i)'\epsilon'_{Y.(W,X')',i})', (1, Y'_i, X'_i)\epsilon_{W.(Y',X')',i},$$
$$(1, Y_{1i}, X'_i)\epsilon_{W.(Y_1,X')',i}, ..., (1, Y_{pi}, X'_i)\epsilon_{W.(Y_p,X')',i}, vec((1, X'_i)'\epsilon'_{Y.X,i})', (1, X'_i)\epsilon_{W.X,i}, vec(\epsilon_{Y.X,i}\epsilon'_{Y.X,i} - \sigma^2_{\tilde{Y}})']'.$$

Recall that Q is finite (by $A_1(i)$) and nonsingular. For a symmetric matrix C and a vector D, let C_1 denote the submatrix that removes the last $\frac{1}{2}p(p+1)$ rows and columns of C and let D_1 be the subvector that removes the last $\frac{1}{2}p(p+1)$ rows of D. Then

$$\sqrt{n}(\tilde{\pi}_1 - \pi^*_1) = \hat{Q}_1^{-1}\sqrt{n}L_1 = (\hat{Q}_1^{-1} - Q_1^{-1})\sqrt{n}L_1 + Q_1^{-1}\sqrt{n}L_1.$$

Since (i) gives $\hat{Q}_1^{-1} - Q_1^{-1} = o_p(1)$ and (ii) gives $\sqrt{n}L_1 \xrightarrow{d} N(0, \Xi_1)$, we obtain that $\sqrt{n}(\tilde{\pi}_1 - \pi^*_1) = Q_1^{-1}\sqrt{n}L_1 + o_p(1) \xrightarrow{d} N(0, \Sigma^*_1)$. Moreover, it follows from $\hat{\mu}^2_{(1,X')'} \xrightarrow{p} \mu^2_{(1,X')'}$, $\sqrt{n}(\tilde{b}_{Y_j.X} - b^*_{Y_j.X}) = O_p(1)$, and $\frac{1}{n}\sum_{i=1}^n \epsilon_{Y_j.X,i}(1, X'_i)' = E[\epsilon_{Y_j.X}(1, X')'] + o_p(1) = o_p(1)$ for $j = 1, ..., p$

that for any $j,h=1,...,p$

$$
\begin{aligned}
&n^{-\frac{1}{2}}\textstyle\sum_{i=1}^{n}\hat{\epsilon}_{Y_j.X,i}\hat{\epsilon}_{Y_h.X,i}\\
&= n^{-\frac{1}{2}}\textstyle\sum_{i=1}^{n}(\epsilon_{Y_j.X,i}-(1,X_i')(\tilde{b}_{Y_j.X}-b^*_{Y_j.X}))(\epsilon_{Y_h.X,i}-(1,X_i')(\tilde{b}_{Y_h.X}-b^*_{Y_h.X}))\\
&= n^{-\frac{1}{2}}\textstyle\sum_{i=1}^{n}\epsilon_{Y_j.X,i}\epsilon_{Y_h.X,i}-[\frac{1}{n}\sum_{i=1}^{n}\epsilon_{Y_j.X,i}(1,X_i')]\sqrt{n}(\tilde{b}_{Y_h.X}-b^*_{Y_h.X})\\
&-[\frac{1}{n}\textstyle\sum_{i=1}^{n}\epsilon_{Y_h.X,i}(1,X_i')]\sqrt{n}(\tilde{b}_{Y_j.X}-b^*_{Y_j.X})+(\tilde{b}_{Y_h.X}-b^*_{Y_h.X})'\hat{\mu}^2_{(1,X')'}\sqrt{n}(\tilde{b}_{Y_j.X}-b^*_{Y_j.X})\\
&= n^{-\frac{1}{2}}\textstyle\sum_{i=1}^{n}\epsilon_{Y_j.X,i}\epsilon_{Y_h.X,i}+o_p(1).
\end{aligned}
$$

Similarly, by (i), we obtain that

$$
\frac{1}{n}\textstyle\sum_{i=1}^{n}\hat{\epsilon}_{Y_j.X,i}\hat{\epsilon}_{Y_h.X,i}=E(\epsilon_{Y_j.X}\epsilon_{Y_h.X})+o_p(1)=\sigma_{\tilde{Y}_j,\tilde{Y}_h}+o_p(1)\quad\text{and}\quad\frac{1}{n}\textstyle\sum_{i=1}^{n}\hat{\epsilon}^2_{W.X,i}=\sigma^2_{\tilde{W}}+o_p(1).
$$

Thus, since $n^{-1/2}\sum_{i=1}^{n}\epsilon_{Y_j.X,i}\epsilon_{Y_h.X,i}$ is $O_p(1)$ by (ii), we have that for $j,h=1,...,p$

$$
\sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_{Y_j.X,i}\hat{\epsilon}_{Y_h.X,i}}{\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}^2_{W.X,i}}=(\sigma^2_{\tilde{W}})^{-1}n^{-\frac{1}{2}}\textstyle\sum_{i=1}^{n}\epsilon_{Y_j.X,i}\epsilon_{Y_h.X,i}+o_p(1).
$$

Together with $\sqrt{n}(\tilde{\pi}_1-\pi_1^*)=Q_1^{-1}\sqrt{n}L_1+o_p(1)$, we obtain by (i) and (ii) that

$$
\sqrt{n}(\tilde{\pi}-\pi^*)=Q^{-1}\sqrt{n}L+o_p(1)\xrightarrow{d}N(0,\Sigma^*)
$$

and therefore that the subvector $\sqrt{n}(\hat{\pi}-\pi)\xrightarrow{d}N(0,\Sigma)$.

Proof of Corollary A.1: The identification set $\mathcal{J}^{k,\tau,\mathbf{c}}$ obtains from A_1-A_6' and the $(Var[(\tilde{Y},\tilde{W})'])$ the moments given by (in)equalities (4-7), using the expressions in Theorem 3.1. The sharpness proof in Corollary 3.2 implies that $\mathcal{J}^{k,\tau,\mathbf{c}}$ is sharp. Specifically, since $G(r)=\sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{\eta}^*}$, we have that $\underline{c}_{jh}\leq r_{\tilde{\eta}_j^*,\tilde{\eta}_h^*}\leq\bar{c}_{jh}$.

To derive $\mathcal{R}^{k,\tau,\mathbf{c}}$, for $j,h=1,...,p$ and $j<h$, consider the restriction

$$
\underline{c}_{jh}\leq\Gamma_{jh}=\frac{G_{jh}(\rho)}{[G_{jj}(\rho)G_{hh}(\rho)]^{\frac{1}{2}}}=\frac{\sigma^{-2}_{\tilde{W}}\sigma_{\tilde{Y}_j,\tilde{Y}_h}-b_{\tilde{Y}_j.\tilde{W}}\frac{1}{\rho}b_{\tilde{Y}_h.\tilde{W}}}{(\sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{Y}_j}-\frac{1}{\rho}b^2_{\tilde{Y}_j.\tilde{W}})^{\frac{1}{2}}(\sigma^{-2}_{\tilde{W}}\sigma^2_{\tilde{Y}_h}-\frac{1}{\rho}b^2_{\tilde{Y}_h.\tilde{W}})^{\frac{1}{2}}}\leq\bar{c}_{jh}.
$$

If $\sigma^2_{\tilde{Y}_j}=0$ or $\sigma^2_{\tilde{Y}_h}=0$ then $\sigma^2_{\eta_j}=0$ or $\sigma^2_{\eta_h}=0$ and $\underline{c}_{jh}\leq\sigma_{\eta_j,\eta_h}\leq\bar{c}_{jh}$ is either incorrect (if $0\notin[\underline{c}_{jh},\bar{c}_{jh}]$) or uninformative about ρ (if $0\in[\underline{c}_{jh},\bar{c}_{jh}]$). Suppose that $\sigma^2_{\tilde{Y}_j}\neq0$ and $\sigma^2_{\tilde{Y}_h}\neq0$. Multiplying the numerator and denominator by $0<\rho\sigma^2_{\tilde{W}}\sigma^{-1}_{\tilde{Y}_j}\sigma^{-1}_{\tilde{Y}_h}$ gives

$$
\underline{c}_{jh}\leq\frac{\rho r_{\tilde{Y}_j,\tilde{Y}_h}-r_{\tilde{W},\tilde{Y}_j}r_{\tilde{W},\tilde{Y}_h}}{(\rho-R^2_{\tilde{W}.\tilde{Y}_j})^{\frac{1}{2}}(\rho-R^2_{\tilde{W}.\tilde{Y}_h})^{\frac{1}{2}}}\leq\bar{c}_{jh}.
$$

The expression for $\mathcal{R}^{\mathbf{c}}_{jh}$ then follows from encoding the sign of r_{η_j,η_h} via the function

$$S_{jh}(r) \equiv r \times r_{\tilde{Y}_j,\tilde{Y}_h} - r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h}$$

and the magnitude of r_{η_j,η_h} ($r^2_{\eta_j,\eta_h} \leq c^2$ or $c^2 \leq r^2_{\eta_j,\eta_h}$) via the quadratic function

$$M_{jh}(r;c) \equiv (r \times r_{\tilde{Y}_j,\tilde{Y}_h} - r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h})^2 - c^2 (r - R^2_{\tilde{W}.\tilde{Y}_j})(r - R^2_{\tilde{W}.\tilde{Y}_h}).$$

By Corollary 3.3, we obtain that $\rho \in \mathcal{R}^{k,\tau,\mathbf{c}} = [R^2_{\tilde{W}.\tilde{Y}}, 1] \cap [\frac{1}{1+\kappa}, 1] \cap_{j=1}^{p} [\frac{1}{\tau_j} R^2_{\tilde{W}.\tilde{Y}_j}, 1] \bigcap_{\substack{j,h=1 \\ j<h}}^{p} \mathcal{R}^{\mathbf{c}}_{jh}$. In addition, $\mathcal{R}^{k,\tau,\mathbf{c}}$ is sharp since every $r \in \mathcal{R}^{k,\tau,\mathbf{c}}$ corresponds to a point $(r,d,b,g) \in \mathcal{J}^{k,\tau,\mathbf{c}}$. Specifically, if $r \in \mathcal{R}^{k,\tau,\mathbf{c}}$ then $\frac{1}{1+\kappa} \leq r \leq 1$, $0 \preceq G(r)$, and $R^2_{\tilde{Y}_j.\tilde{U}} \leq \tau_j$ for $j = 1,...,p$ by Corollary 3.3. Further, since $r \in \mathcal{R}^{k,\tau,\mathbf{c}} \subseteq \mathcal{R}^{\mathbf{c}}_{jh}$, from the sign and magnitude restrictions in $S_{jh}(r)$ and $M_{jh}(r;c)$, we have that $\underline{c}_{jh} \leq \frac{G_{jh}(r)}{[G_{jj}(r)G_{hh}(r)]^{\frac{1}{2}}} \leq \bar{c}_{jh}$ for $j,h = 1,...,p$ with $j < h$.

Next, we examine the behavior of $S_{jh}(r)$ and $M_{jh}(r;c)$ when $\sigma^2_{\tilde{Y}_j}\sigma^2_{\tilde{Y}_h} \neq 0$. First, we have

$$0 \leq S_{jh}(r) \Leftrightarrow \begin{cases} \frac{r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h}}{r_{\tilde{Y}_j,\tilde{Y}_h}} \leq r & \text{when } 0 < r_{\tilde{Y}_j,\tilde{Y}_h} \\ r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h} \leq 0 & \text{when } r_{\tilde{Y}_j,\tilde{Y}_h} = 0 \\ r \leq \frac{r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h}}{r_{\tilde{Y}_j,\tilde{Y}_h}} & \text{when } r_{\tilde{Y}_j,\tilde{Y}_h} < 0 \end{cases}.$$

Further, if $R^2_{\tilde{Y}_j.\tilde{Y}_h} = 1$ then

$$M_{jh}(r;c) = (1-c^2)(r - R^2_{\tilde{W}.\tilde{Y}_j})^2 \geq 0.$$

Suppose instead that $R^2_{\tilde{Y}_j.\tilde{Y}_h} \neq 1$. We obtain

$$\begin{aligned} M_{jh}(r;c) &= r^2 R^2_{\tilde{Y}_j.\tilde{Y}_h} + R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} - 2r \times r_{\tilde{Y}_j,\tilde{Y}_h} r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h} \\ &\quad - c^2 r^2 + c^2 r (R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h}) - c^2 R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} \\ &= r^2 (R^2_{\tilde{Y}_j.\tilde{Y}_h} - c^2) + r[-2 r_{\tilde{Y}_j,\tilde{Y}_h} r_{\tilde{W},\tilde{Y}_j} r_{\tilde{W},\tilde{Y}_h} + c^2 (R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})] + (1-c^2) R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} \\ &= r^2 (R^2_{\tilde{Y}_j.\tilde{Y}_h} - c^2) + r[R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) - (1-c^2)(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})] \\ &\quad + (1-c^2) R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h}, \end{aligned}$$

where the last equality makes use of

$$R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} = \begin{bmatrix} r_{\tilde{W},\tilde{Y}_j} & r_{\tilde{W},\tilde{Y}_h} \end{bmatrix} \begin{bmatrix} 1 & r_{\tilde{Y}_j,\tilde{Y}_h} \\ r_{\tilde{Y}_h,\tilde{Y}_j} & 1 \end{bmatrix}^{-1} \begin{bmatrix} r_{\tilde{W},\tilde{Y}_j} \\ r_{\tilde{W},\tilde{Y}_h} \end{bmatrix} = \frac{R^2_{\tilde{W},\tilde{Y}_j} + R^2_{\tilde{W},\tilde{Y}_h} - 2 r_{\tilde{W},\tilde{Y}_j} r_{\tilde{Y}_j,\tilde{Y}_h} r_{\tilde{W},\tilde{Y}_h}}{1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}}.$$

If $c^2 = R^2_{\tilde{Y}_j.\tilde{Y}_h}$ then $M_{jh}(\cdot;c)$ is a linear function

$$\begin{aligned} M_{jh}(r; r_{\tilde{Y}_j,\tilde{Y}_h}) &= r[R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) - (1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})] \\ &\quad + (1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} \\ &= (1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})\{r[R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} - (R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})] + R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h}\} \end{aligned}$$

and

$$0 \leq M_{jh}(r;c) \Leftrightarrow$$
$$\begin{cases} r \leq \frac{-R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h}}{R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} - (R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})} & \text{when } c^2 = R^2_{\tilde{Y}_j.\tilde{Y}_h} \text{ and } R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} < R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h} \\ 0 \leq (1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} & \text{when } c^2 = R^2_{\tilde{Y}_j.\tilde{Y}_h} \text{ and } R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} = R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h} \\ \frac{-R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h}}{R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} - (R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})} \leq r & \text{when } c^2 = R^2_{\tilde{Y}_j.\tilde{Y}_h} \text{ and } R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h} < R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} \end{cases}$$

Otherwise, if $c^2 \neq R^2_{\tilde{Y}_j.\tilde{Y}_h}$, the discriminant of $M_{jh}(\cdot;c)$ is

$$\begin{aligned} \Delta_{jh}(c) &= [R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) - (1 - c^2)(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})]^2 - 4(1 - c^2)(R^2_{\tilde{Y}_j.\tilde{Y}_h} - c^2)R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} \\ &= [R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) - (1 - c^2)(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})]^2 \\ &- (1 - c^2)4R^2_{\tilde{Y}_j.\tilde{Y}_h} R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} + 4c^2(1 - c^2)R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} \\ &= [R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) - (1 - c^2)(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})]^2 \\ &- (1 - c^2)[R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) - (R^2_{\tilde{W}.\tilde{Y}_h} + R^2_{\tilde{W}.\tilde{Y}_j})]^2 + 4c^2(1 - c^2)R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} \\ &= c^2 R^4_{\tilde{W}.(\tilde{Y}_j.\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})^2 - c^2(1 - c^2)(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})^2 + 4c^2(1 - c^2)R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} \\ &= c^2\{R^4_{\tilde{W}.(\tilde{Y}_j.\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})^2 - (1 - c^2)[(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})^2 - 4R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h}]\} \\ &= c^2[R^4_{\tilde{W}.(\tilde{Y}_j.\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})^2 - (1 - c^2)(R^2_{\tilde{W}.\tilde{Y}_j} - R^2_{\tilde{W}.\tilde{Y}_h})^2]. \end{aligned}$$

In particular, $\Delta_{jh}(c) < 0$ if and only if

$$0 < c^2 < 1 - \frac{R^4_{\tilde{W}.(\tilde{Y}_j.\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})^2}{(R^2_{\tilde{W}.\tilde{Y}_j} - R^2_{\tilde{W}.\tilde{Y}_h})^2}.$$

Further, we have that $1 - \frac{R^4_{\tilde{W}.(\tilde{Y}_j.\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})^2}{(R^2_{\tilde{W}.\tilde{Y}_j} - R^2_{\tilde{W}.\tilde{Y}_h})^2} \leq R^2_{\tilde{Y}_j.\tilde{Y}_h}$ since if $c^2 = R^2_{\tilde{Y}_j.\tilde{Y}_h}$ then

$$\begin{aligned} \Delta_{jh}(c) &= [R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) - (1 - c^2)(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})]^2 - 4(1 - c^2)(R^2_{\tilde{Y}_j.\tilde{Y}_h} - c^2)R^2_{\tilde{W}.\tilde{Y}_j} R^2_{\tilde{W}.\tilde{Y}_h} \\ &= (1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})^2[R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'} - (R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h})]^2 \geq 0 \end{aligned}$$

and if $R^2_{\tilde{Y}_j.\tilde{Y}_h} = 0$ then

$$1 - \frac{R^4_{\tilde{W}.(\tilde{Y}_j.\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h})^2}{(R^2_{\tilde{W}.\tilde{Y}_j} - R^2_{\tilde{W}.\tilde{Y}_h})^2} = 1 - \frac{(R^2_{\tilde{W},\tilde{Y}_j} + R^2_{\tilde{W},\tilde{Y}_h})^2}{(R^2_{\tilde{W}.\tilde{Y}_j} - R^2_{\tilde{W}.\tilde{Y}_h})^2} \leq 0 = R^2_{\tilde{Y}_j.\tilde{Y}_h}.$$

It follows that if $0 < c^2 < 1 - \frac{R^4_{\tilde{W}.(\tilde{Y}_j.\tilde{Y}_h)'}(1-R^2_{\tilde{Y}_j.\tilde{Y}_h})^2}{(R^2_{\tilde{W}.\tilde{Y}_j} - R^2_{\tilde{W}.\tilde{Y}_h})^2}$ then $c^2 < R^2_{\tilde{Y}_j.\tilde{Y}_h}$ and

$$0 \leq M_{jh}(r; c) \Leftrightarrow -\infty < r < \infty.$$

If $c^2 \neq R^2_{\tilde{Y}_j.\tilde{Y}_h}$ and $0 \leq \Delta_{jh}(c)$ then define

$$F_{jh}(c) \equiv -R^2_{\tilde{W}.(\tilde{Y}_j,\tilde{Y}_h)'}(1 - R^2_{\tilde{Y}_j.\tilde{Y}_h}) + (1 - c^2)(R^2_{\tilde{W}.\tilde{Y}_j} + R^2_{\tilde{W}.\tilde{Y}_h}),$$

so that $M_{jh}(\rho; c)$ has the two roots

$$\rho^-_{jh}(c) \equiv \frac{F_{jh}(c) - \Delta_{jh}(c)^{\frac{1}{2}}}{2(R^2_{\tilde{Y}_j.\tilde{Y}_h} - c^2)} \quad \text{and} \quad \rho^+_{jh}(c) \equiv \frac{F_{jh}(c) + \Delta_{jh}(c)^{\frac{1}{2}}}{2(R^2_{\tilde{Y}_j.\tilde{Y}_h} - c^2)}.$$

We then have that

$$0 \leq M_{jh}(r; c) \Leftrightarrow \begin{cases} r \in (-\infty, \rho^-_{jh}(c)] \cup [\rho^+_{jh}(c), \infty) & \text{when } c^2 < R^2_{\tilde{Y}_j.\tilde{Y}_h} \\ r \in [\rho^+_{jh}(c), \rho^-_{jh}(c)] & \text{when } R^2_{\tilde{Y}_j.\tilde{Y}_h} < c^2 \end{cases}.$$

Combining these results, yields the equivalence between $0 \leq M_{jh}(r; c)$ and the range of r.

The sharp bounds $\mathcal{D}^{k,\tau,\mathbf{c}}$, $\mathcal{B}^{k,\tau,\mathbf{c}}$, and $\mathcal{G}^{k,\tau,\mathbf{c}}$ follow from the mappings $D(\cdot)$, $B(\cdot)$, and $G(\cdot)$ in Theorem 3.1.

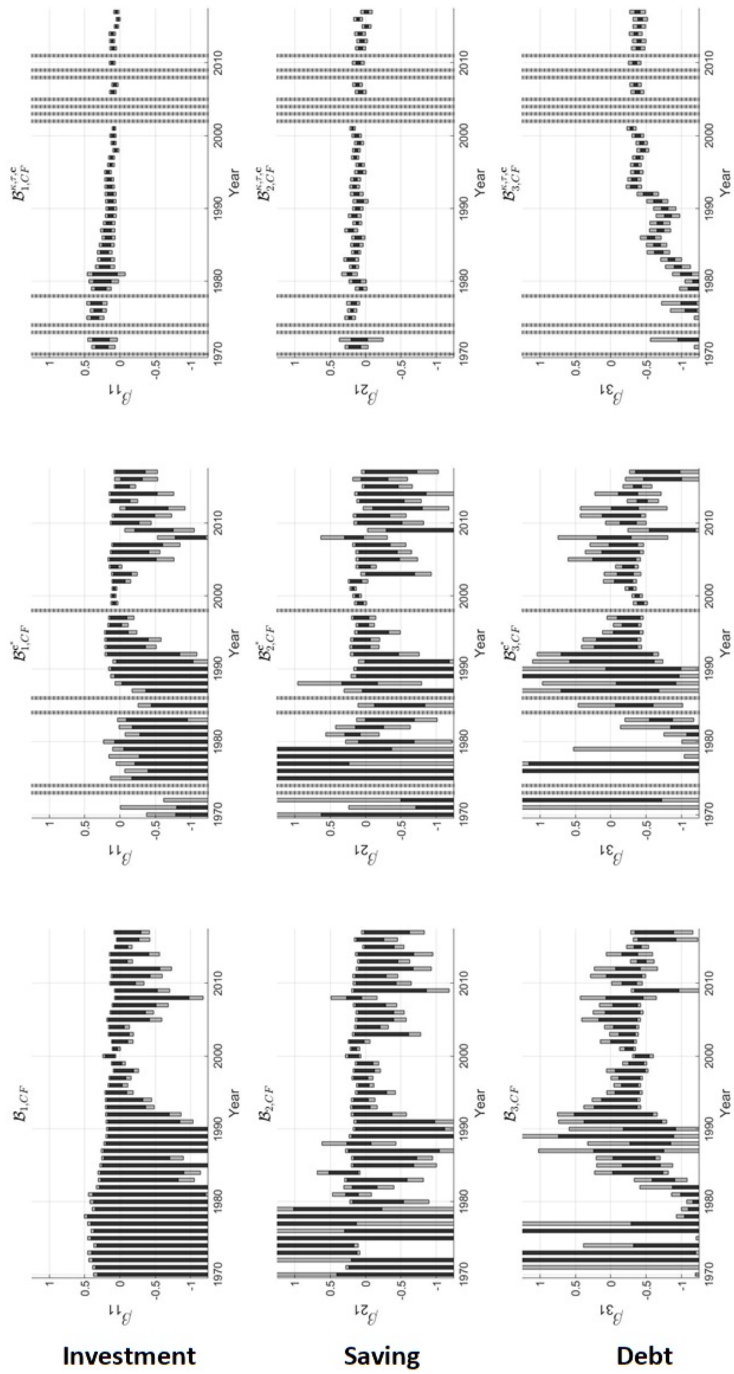

Figure 4: 50% (dark) and 95% (light) confidence regions for β_{j1} (cash flow) for $j = 1, 2, 3$ (investment, saving, and debt) from year 1970 to 2017, when X includes asset tangibility. We consider the regions $\mathcal{B}_{j1}$, $\mathcal{B}_{j1}^{\mathbf{c}^*}$, and $\mathcal{B}_{j1}^{\kappa,\tau,\mathbf{c}}$ where $\mathbf{c}^* = 0$, κ and τ are such that $\hat{\kappa}^* = 0.5$ and $\hat{\tau}^* = (0.9, 0.9, 0.9)'$, and $\mathbf{c}$ is such that $(\underline{c}_{12}, \overline{c}_{12}) = (\underline{c}_{23}, \overline{c}_{23}) = (-1, 0)$ and $(\underline{c}_{13}, \overline{c}_{13}) = (0, 1)$. The shaded vertical bars indicate years in which the maintained assumptions are rejected.

Table 7: Bounds on the Cash Flow Coefficients in the Investment, Saving, and Debt Equations Using the Full Panel and Accounting for Asset Tangibility

	$\mathcal{S}_j^{\kappa,\tau}$	$\mathcal{J}^{\kappa,\tau}$	$\mathcal{J}^{\kappa,\tau,\mathbf{c}}$	$\mathcal{J}^{\kappa,\tau,\mathbf{c}^*}$	$b_{Y.(W,X')'}$
	Results without fixed effects for $\kappa = \infty$ and $\tau = (1,1,1)'$				
β_{11}	[-0.172 , 0.145]	[-0.049 , 0.145]	[0.118 ,0.146]	[0.118, 0.141]	0.143
	(-0.213 , 0.151)	(-0.074 , 0.151)	(0.111 , 0.152)	(0.110, 0.148)	(0.136 , 0.150)
β_{21}	[-0.540 , 0.124]	[-0.029 , 0.124]	[0.102 , 0.124]	[0.101, 0.121]	0.121
	(-0.621 , 0.130)	(-0.048 , 0.130)	(0.094 , 0.131)	(0.094, 0.128)	(0.114 , 0.129)
β_{31}	[-0.424 , 1.865]	[-0.424 , -0.224]	[-0.425 ,-0.390]	[-0.421, -0.389]	-0.418
	(-0.438 , 2.229)	(-0.438 , -0.189)	(-0.442 , -0.372)	(-0.438, -0.371)	(-0.436 , -0.400)
	Results without fixed effects for $\hat{\kappa}^* = 0.5$ and $\hat{\tau}^* = (0.9, 0.9, 0.9)'$				
β_{11}	[0.125 , 0.145]	[0.125 , 0.145]	[0.124 ,0.146]	[0.124, 0.141]	0.143
	(0.119 , 0.151)	(0.119 , 0.151)	(0.117 , 0.152)	(0.117, 0.148)	(0.136 , 0.150)
β_{21}	[0.106 , 0.124]	[0.106 , 0.124]	[0.106 , 0.124]	[0.106, 0.121]	0.121
	(0.101 , 0.130)	(0.101 , 0.130)	(0.099 , 0.131)	(0.099, 0.128)	(0.114 , 0.129)
β_{31}	[-0.424 , -0.396]	[-0.424 , -0.396]	[-0.425 ,-0.395]	[-0.421,-0.395]	-0.418
	(-0.438 , -0.382)	(-0.438 , -0.382)	(-0.442 , -0.378)	(-0.438, -0.378)	(-0.436 , -0.400)
	Results with year and firm fixed effects for $\kappa = \infty$ and $\tau = (1,1,1)'$				
β_{11}	[-0.719 , 0.132]	[-0.549 , 0.132]	[-0.565 ,0.132]	[-0.564, -0.154]	0.129
	(-0.758 , 0.138)	(-0.581 , 0.138)	(-0.606 , 0.139)	(-0.605,-0.136)	(0.122 , 0.137)
β_{21}	[-2.913 , 0.174]	[-0.329 , 0.174]	[-0.342 ,0.174]	[-0.341, -0.031]	0.170
	(-3.142 , 0.182)	(-0.369 , 0.182)	(-0.392 , 0.184)	(-0.391, -0.006)	(0.159 , 0.181)
β_{31}	[-0.374 , ∞]	[-0.374 , -0.288]	[-0.374 , -0.285]	[-0.357, -0.285]	-0.368
	($-\infty$, ∞)	(-0.388 , -0.237)	(-0.402 , -0.219)	(-0.402, -0.220)	(-0.383 , -0.353)
	Results with year and firm fixed effects for $\hat{\kappa}^* = 0.5$ and $\hat{\tau}^* = (0.9, 0.9, 0.9)'$				
β_{11}	[0.081 , 0.132]	[0.081 , 0.132]	[0.081 , 0.132]	-	0.129
	(0.075 , 0.138)	(0.075 , 0.138)	(0.073 , 0.139)	-	(0.122 , 0.137)
β_{21}	[0.133 , 0.174]	[0.133 , 0.174]	[0.132 , 0.174]	-	0.170
	(0.124 , 0.182)	(0.124 , 0.182)	(0.121 , 0.184)	-	(0.159 , 0.181)
β_{31}	[-0.374 , -0.358]	[-0.374 , -0.358]	[-0.374 ,-0.358]	-	-0.368
	(-0.385 , -0.346)	(-0.385 , -0.346)	(-0.388 , -0.342)	-	(-0.383 , -0.353)

The sample is an unbalanced panel of 161,959 firm-year observations. Y_1, Y_2, and Y_3 denote Investment, Saving, and Debt respectively and $X = [\text{Cash Flow}, \text{Firm Size}, \text{Asset Tangibility}]$. When year fixed effects are included, X also includes year indicator variables. When firm fixed effects are included, the equations' variables undergo a within transformation. $\mathbf{c}$ sets $(\underline{c}_{12}, \overline{c}_{12}) = (\underline{c}_{23}, \overline{c}_{23}) = (-1, 0)$ and $(\underline{c}_{13}, \overline{c}_{13}) = (0, 1)$ whereas $\mathbf{c}^* = 0$. Robust standard errors for π are clustered by firm. 50% and 95% confidence regions are in brackets and parentheses respectively.